Input–to–state Stability

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Part IV: Applications
Consider

\[ \dot{x}(t) = f(x(t), w(t)) \]

with solutions \( \varphi(t, x, w) \)

The system is called ISS, if there exist \( \beta \in K\mathcal{L} \) and \( \gamma \in K_{\infty} \) such that for all initial values \( x \), all perturbation functions \( w \) and all times \( t \geq 0 \) the following inequality holds:

\[ \| \varphi(t, x, w) \| \leq \max \{ \beta(\|x\|, t), \gamma(\|w\|_{\infty}) \} \]

In this part of the course, we will investigate applications and some aspects of ISS controller design

Applications: stability of interconnected and discretized systems
GAS Cascades

Consider

\[
\begin{align*}
\dot{x}_1(t) &= f_1(x_1(t)) \\
\dot{x}_2(t) &= f_2(x_2(t), w(t))
\end{align*}
\]
GAS Cascades

**Coupling** via $w = x_1$ yields

\[
\begin{align*}
\dot{x}_1(t) &= f_1(x_1(t)) \\
\dot{x}_2(t) &= f_2(x_2(t), x_1(t))
\end{align*}
\]

**Theorem:** $x_1$ GAS + $x_2$ ISS $\Rightarrow$ coupled system GAS
GAS Cascades

**Proof:** The proof is easier with ISDS formulation

\[
\|\varphi_1(t, x_1^0)\| \leq \mu_1(\sigma_1(\|x_1^0\|), t))
\]

\[
\|\varphi_2(t, x_2^0, w)\| \leq \max\{\mu_2(\sigma_2(\|x_2^0\|), t), \nu_2(w, t)\}
\]

\[
\nu_2(w, t) := \text{ess sup}_{\tau \in [0, t]} \mu_2(\gamma_2(\|w(\tau)\|), t - \tau)
\]

For \( w = \varphi_1 \) we obtain

\[
\nu_2(\varphi_1, t) \leq \max_{\tau \in [0, t]} \mu_2(\gamma_2(\mu_1(\sigma_1(\|x_1^0\|), \tau), t - \tau) =: \beta_2(\|x_1^0\|, t) \in \mathcal{KL}
\]

Thus \( \varphi = (\varphi_1, \varphi_2) \) with \( \|\varphi\| = \max\{\|\varphi_1\|, \|\varphi_2\|\} \) satisfies

\[
\|\varphi(t, x^0)\| \leq \max\{\mu_1(\sigma_1(\|x_1^0\|), t)), \mu_2(\sigma_2(\|x_2^0\|), t), \beta_2(\|x_1^0\|, t)\}
\]
GAS Cascades

**Note:** The function

\[ \beta_2(\|x^0_1\|, t) := \max_{\tau \in [0,t]} \mu_2(\gamma_2(\mu_1(\sigma_1(\|x^0_1\|), \tau), t - \tau) \]

takes care of the coupling

For

\[ \dot{\mu}_1 = -g_1(\mu_1) \quad \text{and} \quad \dot{\mu}_2 = -g_2(\mu_2) \]

it is bounded by

\[ \beta_2(r, t) \leq \eta_2(\gamma_2 \circ \sigma_1(r), t) \]

with \( \eta_2 \in KLD \) given by

\[ \dot{\eta}_2 = \max\{-g_2(\eta_2), -\gamma'_2(\gamma_2^{-1}(\eta_2)) g_1(\gamma_2^{-1}(\eta_2))\} \]
Small Gain Theorem

Consider

\[
\begin{align*}
\dot{x}_1(t) &= f_1(x_1(t), w_1(t)) \\
\dot{x}_2(t) &= f_2(x_2(t), w_2(t))
\end{align*}
\]
Small Gain Theorem

Coupling via $w_1 = x_2, w_2 = x_1$ yields

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t))$$
$$\dot{x}_2(t) = f_2(x_2(t), x_1(t))$$

Theorem: $x_1$ ISS + $x_2$ ISS + $\gamma_1(\gamma_2(r)) < r$  
$\implies$ coupled system GAS
Small Gain Theorem

Proof: For $\theta < 1$ we use ISDS in order to analyze

$$\dot{x}_1(t) = f_1(x_1(t), \theta x_2(t)), \quad \dot{x}_2(t) = f_2(x_2(t), \theta x_1(t))$$

Induction over a suitable time sequence yields GAS with

$$\|\varphi_i(t, x^0)\| \leq \eta_i(\alpha_i(\|x^0\|), t)$$

where

$$\alpha_i(x) \leq \max\{\sigma_i(\|x_i\|), \gamma_i \circ \sigma_j(\|x_j\|)\}$$

with $\eta_i \in \mathcal{KL}$ determined by

$$\dot{\eta}_i = \max\{-g_i(\eta_i), -\gamma_i'\gamma_i^{-1}(\eta_i) g_j(\gamma_i^{-1}(\eta_i))\}$$

$i = 1, 2, j = 2, 1$

By continuity, the estimate also holds for $\theta = 1$
\[ \dot{x}_1 = -x_1 + \frac{w_1^3}{2} \quad \text{and} \quad \dot{x}_2 = -x_2^3 + w_2 \]

Using \( V_1(x) = V_2(x) = |x| \) we obtain

\[
\begin{align*}
\mu_1(r, t) &= e^{-t/4}r, \quad \gamma_1(r) = 2r^3/2 \\
\mu_2(r, t) &= \sqrt{\frac{2t+4}{r^2}} t + \frac{2}{r^2}, \quad \gamma_2(r) = \frac{3}{4r^{3/3}}
\end{align*}
\]

\( \Rightarrow \) the coupled system is asymptotically stable with

\[ |\varphi_i(t, x_0)| \leq \eta_i(\alpha_i(x_0), t), \] where

\[
\begin{align*}
\alpha_1(x) &= \max \left\{ |x_1|, \frac{2}{3}|x_2|^3 \right\}, \quad \dot{\eta}_1 = \max\{-c_1\eta_1, -c_2\eta_1^{5/3}\} \\
\alpha_2(x) &= \max \left\{ |x_2|, \frac{3}{\sqrt{3}}|x_1| \right\}, \quad \dot{\eta}_2 = \max\{-c_3\eta_2, -c_4\eta_2^3\}
\end{align*}
\]
Small Gain Theorem — ISS version
([Jiang/Teel/Praly 94, Teel 95])

Consider

$$\dot{x}_1(t) = f_1(x_1(t), w_1(t), v_1(t))$$

$$\dot{x}_2(t) = f_2(x_2(t), w_2(t), v_2(t))$$

write ISS as

$$\|\varphi_i(t, x, w_i, v_i)\| \leq \max\{\beta_i(\|x\|, t), \gamma_{w_i}(\|w_i\|_{\infty}), \gamma_{v_i}(\|v_i\|_{\infty})\}$$
Small Gain Theorem — ISS version
([Jiang/Teel/Praly 94, Teel 95])

Coupling via $w_1 = x_2$, $w_2 = x_1$ yields

\[
\begin{align*}
\dot{x}_1(t) & = f_1(x_1(t), x_2(t), v_1(t)) \\
\dot{x}_2(t) & = f_2(x_2(t), x_1(t), v_2(t))
\end{align*}
\]

\textbf{Theorem:} \quad x_1 \& x_2 \text{ ISS} + \gamma_{w_1}(\gamma_{w_2}(r)) < r

$\implies$ coupled system ISS
Small Gain Theorems

Recall the notion of input-to-output stability (IOS) for systems with output $y = h(x)$:

$$\|y(t)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\}$$

Using IOS, the small gain results are easily extended to systems with output.
Numerical Discretization

All stability concepts are easily extended to compact sets $A \subset \mathbb{R}^n$

Denoting the Euclidean distance of $x \in \mathbb{R}^n$ to $A$ by $\|x\|_A$ we can, e.g., define

$A$ is called (locally) **asymptotically stable** with **neighborhood** $B$ and **attraction rate** $\beta \in \mathcal{KL}$, if for all $x \in B$

$$\|\phi(t, x)\|_A \leq \beta(\|x\|_A, t), \quad t \geq 0$$

Similarly, all the **ISS concepts** can be generalized
Numerical Discretization

**Goal:** find asymptotically stable sets by *numerical simulations*, e.g., using a one step method

\[ x(t + h) = \varphi_h(x(t)), \]

with *solution trajectories* \( \varphi_h(t, x_0) \)
Theorem [Kloeden/Lorenz 86] Let $A$ be an asymptotically stable set for $\varphi(t, x)$ and let be $\tilde{\varphi}_h$ an approximation of $\varphi$ by a numerical one–step method with

$$\|\tilde{\varphi}_h(h, x) - \varphi(h, x)\| \leq ch^{q+1}$$

Then $\tilde{\varphi}_h$ has “numerical” asymptotically stable sets $\tilde{A}_h$ with Hausdorff limit $\lim_{h \to 0} \tilde{A}_h = A$.

**But:** For arbitrary numerical as. stable sets $\tilde{A}_h$ the limit $A = \lim_{h \to 0} \tilde{A}_h$ is not asymptotically stable for $\varphi$.

**Question:** When is $A = \lim_{h \to 0} \tilde{A}_h$ as. stable for $\varphi$?

**Idea:** Interpret $\tilde{\varphi}_h$ as perturbed system $\dot{x} = f(x) + w$.
Numerical Discretization

**Theorem:** \( A = \lim_{h \to 0} \tilde{A}_h \) asymptotically stable

\[ \iff \]

the sets \( \tilde{A}_h \) are locally ISS with \( \beta_h \in \mathcal{KL}, \gamma_h \in \mathcal{K}_\infty \) such that

\[
\beta_h \to \beta \in \mathcal{KL} \quad \text{and} \quad \gamma_h \to \gamma \in \mathcal{K}_\infty \quad \text{for} \quad h \to 0
\]

\[ \iff \]

the sets \( \tilde{A}_h \) have attraction rates \( \beta_h \in \mathcal{KL} \) with

\[
\beta_h \to \beta \in \mathcal{KL} \quad \text{for} \quad h \to 0
\]
Numerical Discretization: Example

For \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \) consider

\[
\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x - \max\{\|x\| - 1, 0\} x
\]

Euler approximation suggests that

\[ S_1 = \{ x \in \mathbb{R}^2 | \|x\| = 1 \} \]

is asymptotically stable

In fact,

\[ D_1 = \{ x \in \mathbb{R}^2 | \|x\| \leq 1 \} \]

is the only asymptotically stable set
Construction of ISS Feedbacks

Consider

\[ \dot{x}(t) = f(x(t), u(t), w(t)) \]
Construction of ISS Feedbacks

Find a feedback law $u$ such that

$$\dot{x}(t) = f(x(t), u(x(t)), w(t))$$

is ISS
Construction of ISS Feedbacks

This is a special case of stabilization via feedback, hence the same obstructions arise:

- continuous static state feedbacks might not exist (Brockett’s condition)
- no “universal” design method for the general nonlinear case

Here, we will focus on

- an abstract existence result
- a design procedure based on an ISS Lyapunov function
Sampled solutions

How to define solutions for a discontinuous feedback map $u : \mathbb{R}^n \to U$? — by sampling:

Consider a sampling sequence

$$\pi = (t_i)_{i \in \mathbb{N}_0}, \quad 0 = t_0 < t_1 < t_2 < \ldots \to \infty$$

with maximal sampling rate

$$\Delta(\pi) := \sup_{i \in \mathbb{N}} t_i - t_{i-1} < \infty$$

Define sampled solution $\varphi_\pi(t, x)$ recursively for $i = 0, 1, 2, \ldots$ via

$$x_i := \varphi_\pi(t_i, x), \quad \varphi_\pi(t, x) := \varphi(t - t_i, x_i, u(x_i)), \quad t \in [t_i, t_{i+1}]$$
Stabilization by discontinuous feedback

This framework allows for a general abstract stabilization result

Theorem [Clarke/Ledyaev/Sontag/Subbotin 97]:

If the system is asymptotically controllable to 0 then there exists a feedback \( u : \mathbb{R}^n \to U \) such that the sampled system is semiglobally practically asymptotically stable, i.e.:

there exists \( \beta \in \mathcal{KL} \) such that for all \( R, \varepsilon > 0 \) there is \( \delta > 0 \) with

\[
\| \varphi_\pi(t, x) \| \leq \beta(\| x \|, t) + \varepsilon
\]

if \( \| x \| \leq R \) and \( \Delta(\pi) \leq \delta \)
ISS Stabilization by discontinuous feedback

Consider the input affine system

\[ \dot{x} = f(x) + G(x)u + G(x)w \]

**Theorem:** If the system is asymptotically controllable to 0 for\( w \equiv 0 \) then there exists a feedback \( u : \mathbb{R}^n \rightarrow U \) such that the sampled system is semiglobally practically ISS, i.e.:

There exists \( \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty \) such that for all \( R, \varepsilon > 0 \) there is \( \delta > 0 \) with

\[ \|\varphi_\pi(t, x, w)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\} + \varepsilon \]

if \( \|x\| \leq R, \gamma(\|w\|_\infty) \leq R \) and \( \Delta(\pi) \leq \delta \)

**Idea of Proof:** Use control Lyapunov function \( V \) and nonsmooth analysis techniques to make \( V \) an ISS Lyapunov function
ISS Stabilization via universal formula

A more constructive approach is based on ISS control Lyapunov functions (ISS clf) for control affine systems \( \dot{x} = f(x, w) + G(x)u \):

A smooth function \( V : \mathbb{R}^n \to \mathbb{R} \) is called an ISS clf (in dissipation form), if there exist \( \alpha_i \in \mathcal{K}_\infty, i = 1, 2, 3, 4 \), such that

\[
\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)
\]

and

\[
\inf_{u \in U} DV(x)(f(x, w) + G(x)u) \leq -\alpha_3(\|x\|) + \alpha_4(\|w\|)
\]

hold for all \( x \in \mathbb{R}^n, w \in \mathbb{R}^m \)
**ISS Stabilization via universal formula**

**Given:** \[ \inf_{u \in U} DV(x)(f(x, w) + G(x)u) \leq -\alpha_3(\|x\|) + \alpha_4(\|w\|) \]

**Theorem:** Consider \( u(x) = K(\bar{\omega}(x), DV(x)G(x)^T) \) with

\[
K(a, b) := \begin{cases} 
-a + \sqrt{a^2 + \|b\|^4} & b \neq 0 \\
0, & b = 0 
\end{cases}
\]

and \( \bar{\omega} \) being a continuous and outside 0 smooth function with

\[
\bar{\omega}(x) \approx \omega(x) := \max_w \{DV(x)f(x, w) - \alpha_4(\|w\|)\}
\]

and assume the small control property, i.e., for small \( \|x\| \) there exists small \( \|u\| \) with \( \omega(x) + DV(x)G(x)u \leq -\alpha_3(\|x\|) \)

Then \( u \) is continuous and smooth outside 0 and the closed loop system \( \dot{x} = f(x, w) + G(x)u(x) \) is ISS (in the classical sense)
ISS Stabilization via universal formula

A similar result is available for integral ISS

Proof: Show that $V$ is an ISS Lyapunov function for the closed loop system
Summary of Part IV

• ISS can be used for the stability analysis of cascades and fully interconnected systems

• ISS can be used for the analysis of numerical discretizations

• ISS controller design: abstract result and universal formula