

Random attractors and the preservation of synchronization in the presence of noise*

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Deterministic case

Consider the ordinary differential equations in \mathbb{R}^d

$$\frac{dx}{dt} = f(x), \quad \frac{dy}{dt} = g(y),$$

- f, g regular and satisfy the one-sided Lipschitz conditions

$$\langle x_1 - x_2, f(x_1) - f(x_2) \rangle \leq -L|x_1 - x_2|^2,$$

$$\langle y_1 - y_2, g(y_1) - g(y_2) \rangle \leq -L|y_1 - y_2|^2,$$

on \mathbb{R}^d for some $L > 0$,

\Rightarrow globally asymptotically stable equilibria \bar{x} and \bar{y} .

Now consider the dissipatively coupled system

$$\frac{dx}{dt} = f(x) + \nu(y - x), \quad \frac{dy}{dt} = g(y) + \nu(x - y)$$

with $\nu > 0$.

\Rightarrow globally asymptotically stable equilibrium $(\bar{x}^\nu, \bar{y}^\nu)$.

$(\bar{x}^\nu, \bar{y}^\nu) \rightarrow (\bar{z}, \bar{z})$ as $\nu \rightarrow \infty$, where \bar{z} is the unique globally asymptotically stable equilibrium of the averaged system

$$\frac{dz}{dt} = \frac{1}{2} (f(z) + g(z)).$$

This is known as synchronization

Stochastic case

What is the effect of environmental noise on synchronization?

Coupled Ito stochastic differential equations with additive noise

$$dX_t = (f(X_t) + \nu(Y_t - X_t)) dt + \alpha dW_t^1,$$

$$dY_t = (g(Y_t) + \nu(X_t - Y_t)) dt + \beta dW_t^2$$

where W_t^1, W_t^2 are independent two-sided scalar Wiener processes and $\alpha, \beta \in \mathbb{R}^d$ are constant vectors.

$\Rightarrow \exists$ unique stochastic stationary solution $(\bar{X}_t^\nu, \bar{Y}_t^\nu)$, which is pathwise globally asymptotically stable.

Moreover on finite time intervals $[T_1, T_2]$ of \mathbb{R}

$$(\bar{X}_t^\nu, \bar{Y}_t^\nu) \rightarrow (\bar{Z}_t^\infty, \bar{Z}_t^\infty) \quad \text{as } \nu \rightarrow \infty \quad \underline{\text{pathwise}}$$

where \bar{Z}_t^∞ is the unique pathwise globally asymptotically stable stationary solution of the averaged SDE

$$dZ_t = \frac{f(Z_t) + g(Z_t)}{2} dt + \frac{\alpha}{2} dW_t^1 + \frac{\beta}{2} dW_t^2.$$

[*Caraballo & Kloeden*, Proc. Roy. Soc. London (2005)]

Recall:

- The solutions of Ito stochastic differential equations are pathwise continuous, but not differentiable.
- Ito SDEs are really stochastic integral equations with stochastic integrals defined in the mean-square or L_2 sense.

How do we apply the Lipschitz properties to obtain pathwise estimates?

A technical detour : Consider the Ito SDE

$$dX_t = f(X_t) dt + \alpha dW_t$$

where f satisfies the one-sided Lipschitz condition.

i.e., the stochastic integral equation

$$X_t = X_{t_0} + \int_{t_0}^t f(X_s) ds + \alpha \int_{t_0}^t dW_t$$

The difference of any two solutions satisfies pathwise

$$X_t^1 - X_t^2 = X_{t_0}^1 - X_{t_0}^2 + \int_{t_0}^t \underbrace{[f(X_s^1) - f(X_s^2)]}_{\text{continuous paths}} ds$$

Fundamental theorem of calculus $\Rightarrow X_t^1 - X_t^2$ pathwise differentiable.

$$\frac{d}{dt} [X_t^1 - X_t^2] = f(X_t^1) - f(X_t^2) \quad \text{pathwise}$$

- Apply the one-sided Lipschitz condition

$$\frac{d}{dt} |X_t^1 - X_t^2|^2 = 2 \langle X_t^1 - X_t^2, f(X_t^1) - f(X_t^2) \rangle \leq -2L |X_t^1 - X_t^2|^2$$

$$\Rightarrow |X_t^1 - X_t^2|^2 \leq |X_{t_0}^1 - X_{t_0}^2|^2 e^{-2L(t-t_0)} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

i.e. all solutions converge pathwise together — but to what?

Special case : Ito SDE with linear drift $f(x) = -x$

$$dX_t = -X_t dt + \alpha dW_t$$

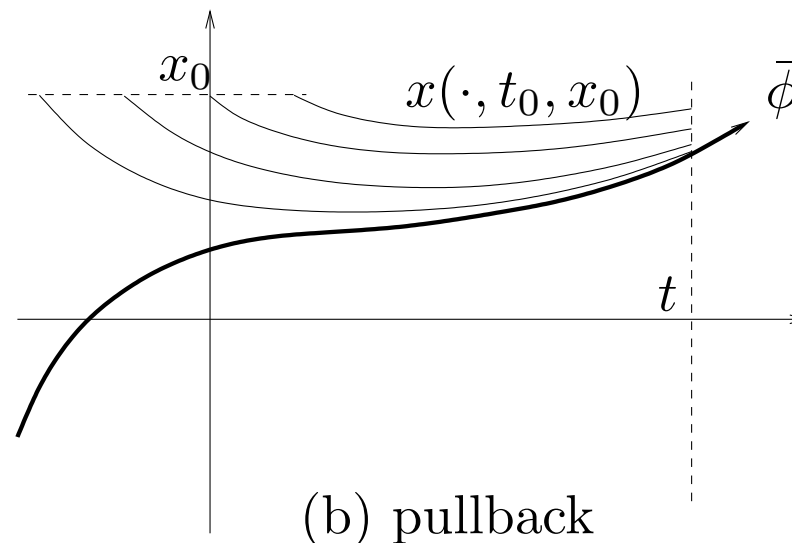
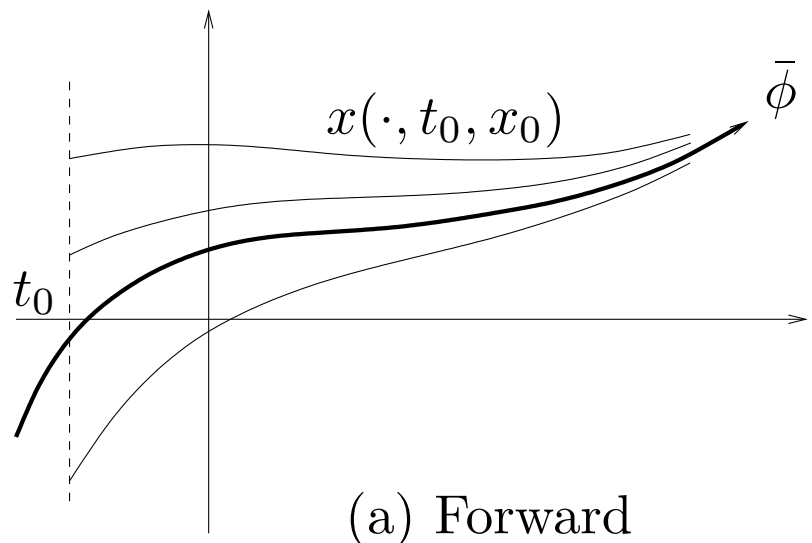
explicit solution

$$X_t = X_{t_0} e^{-(t-t_0)} + \alpha e^{-t} \int_{t_0}^t e^s dW_s$$

The forward limit as $t \rightarrow \infty$ does not exist — moving target!

But the pullback limit as $t_0 \rightarrow -\infty$ with t fixed does exist:

$$\lim_{t_0 \rightarrow -\infty} X_t = \bar{O}_t := \alpha e^{-t} \int_{-\infty}^t e^s dW_s \quad (\text{pathwise})$$



The Ornstein-Uhlenbeck stochastic stationary process \bar{O}_t is a solution of the linear SDE and all other solutions converge pathwise to it in the forward sense

$$|X_t - \bar{O}_t| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (\text{pathwise})$$

Random dynamical systems : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (X, d_X) a metric space.

A random dynamical system (θ, ϕ) on $\Omega \times X$ consists of

- a metric dynamical system θ on Ω , which models the noise,
- a cocycle mapping $\phi : \mathbb{R}^+ \times \Omega \times X \rightarrow X$, which represents the dynamics on the state space X and satisfies

- 1). $\phi(0, \omega, x_0) = x_0$ (initial condition)
- 2). $\phi(s + t, \omega, x_0) = \phi(s, \theta_t \omega, \phi(t, \omega, x_0))$ (cocycle property)
- 3). $(t, x_0) \mapsto \phi(t, \omega, x_0)$ is continuous (continuity)
- 4). $\omega \mapsto \phi(t, \omega, x_0)$ is \mathcal{F} -measurable (measurability)

for all $s, t \geq 0$, $x_0 \in X$ and $\omega \in \Omega$.

Random attractors

A random attractor is a family of nonempty measurable compact subsets of X

$$\widehat{A} = \{A(\omega) : \omega \in \Omega\}$$

which is

- ϕ -invariant $\phi(t, \omega, A(\omega)) = A(\theta_t \omega)$ for all $t \geq 0$,
- pathwise pullback attracting in the sense that

$$\text{dist}_X (\phi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), A(\omega)) \rightarrow 0 \quad \text{for } t \rightarrow +\infty$$

for all suitable families $\widehat{D} = \{D(\omega) : \omega \in \Omega\}$ of nonempty measurable bounded subsets of X .

Theorem (Crauel, Flandoli, Schmalfuß etc)

Let (θ, ϕ) be an RDS on $\Omega \times X$ such that $\phi(t, \omega, \cdot) : X \rightarrow X$ is a compact operator for each fixed $t > 0$ and $\omega \in \Omega$.

If there exists a pullback absorbing family $\widehat{B} = \{B(\omega) : \omega \in \Omega\}$ of nonempty closed and bounded measurable subsets of X , i.e. there exists a $T_{\widehat{D}, \omega} \geq 0$ such that

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega) \quad \text{for all } t \geq T_{\widehat{D}, \omega}$$

for all $\widehat{D} = \{D(\omega) : \omega \in \Omega\}$ in a given attracting universe.

Then the RDS $\Omega \times X$ has a random attractor \widehat{A} with component subsets given by

$$A(\omega) = \bigcap_{s>0} \overline{\bigcup_{t \geq s} \phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))} \quad \text{for each } \omega \in \Omega.$$

General case again

Subtract the integral version of the linear SDE for \bar{O}_t from the integral version of the nonlinear SDE

$$dX_t = f(X_t) dt + \alpha dW_t$$

to obtain

$$X_t - \bar{O}_t = X_{t_0} - \bar{O}_{t_0} + \int_{t_0}^t [f(X_s) - \bar{O}_s] ds$$

$\Rightarrow V_t := X_t - \bar{O}_t$ is pathwise differentiable and satisfies the pathwise ODE

$$\frac{d}{dt} V_t = f(V_t + \bar{O}_t) - \bar{O}_t \quad (\text{pathwise})$$

- Apply the one-sided Lipschitz condition pathwise to

$$\frac{d}{dt} [X_t - \bar{O}_t] = [f(X_t) - f(\bar{O}_t)] + [f(\bar{O}_t) + \bar{O}_t] \quad (\text{pathwise})$$

to obtain the pathwise estimate

$$|V_t|^2 \leq |V_{t_0}|^2 e^{-L(t-t_0)} + \frac{2}{L} e^{-Lt} \int_{t_0}^t e^{Ls} (|f(\bar{O}_s)|^2 + |\bar{O}_s|^2) ds$$

- Take pathwise pullback convergence as $t_0 \rightarrow -\infty$ to obtain

$$|X_t - \bar{O}_t| \leq \bar{R}_t := 1 + \frac{2}{L} e^{-Lt} \int_{-\infty}^t e^{Ls} (|f(\bar{O}_s)|^2 + |\bar{O}_s|^2) ds$$

for $t \geq T$ depending on suitable bounded sets of initial values.

- i.e., there exists a family of compact pullback absorbing balls centered on \bar{O}_t with random radius \bar{R}_t .
 - Dynamical systems limit set ideas
 - \Rightarrow there exists a compact setvalued stochastic process A_t inside these absorbing balls which pathwise pullback attracts the solutions.
 - BUT the solutions converge together pathwise in forwards sense, so the sets A_t are in fact all singleton sets
- $\Rightarrow \exists$ stochastic stationary solution \bar{X}_t .

General Principles

- All regular Ito SDE in \mathbb{R}^d can be transformed into pathwise ODE

[*Imkeller & Schmalfuß* (2001), *Imkeller & Lederer* (2001,2002)]

- and generate random dynamical systems

⇒ **pathwise theory and numerics for Ito SDE**

- Pullback convergence enables us to construct moving targets.

- Stochastic stationary solutions are a simple singleton set version of more general random attractors

⇒ **theory of random dynamical systems**

e.g., Ludwig Arnold (Bremen)

- parallel theory of **deterministic skew product flows**

e.g., almost periodic ODE : George Sell (Minneapolis)

⇒ **A theory of nonautonomous dynamical systems**

e.g., pullback attractors

Effects of discretization on synchronization

Numerical Ornstein-Uhlenbeck process

For the linear SDE with additive noise,

$$dX_t = -X_t dt + \alpha dW_t,$$

the drift-implicit Euler-Maruyama scheme with constant step size $h > 0$ is

$$X_{n+1} = X_n - hX_{n+1} + \alpha\Delta W_n, \quad n = n_0, n_0 + 1, \dots,$$

which simplifies algebraically to

$$X_{n+1} = \frac{1}{1+h}X_n + \frac{\alpha}{1+h}\Delta W_n,$$

Here the $\Delta W_n = W_{h(n+1)} - W_{hn}$ are mutually independent and $N(0, h)$ distributed

It follows that

$$X_n = \frac{1}{(1+h)^{n-n_0}} X_{n_0} + \frac{\alpha}{1+h} \sum_{j=n_0}^{n-1} \frac{1}{(1+h)^{n-1-j}} \Delta W_j$$

and the pathwise pullback limit, i.e. with n fixed and $n_0 \rightarrow -\infty$, gives the discrete time numerical Ornstein-Uhlenbeck process

$$\widehat{O}_n^{(h)} := \frac{\alpha}{1+h} \sum_{j=-\infty}^{n-1} \frac{1}{(1+h)^{n-1-j}} \Delta W_j, \quad n \in \mathbb{Z}. \quad (1)$$

which is an entire solution of the numerical scheme and a discrete time stochastic stationary process.

One can show that it converges pathwise to the continuous time Ornstein-Uhlenbeck process in the sense that

$$\widehat{O}_0^{(h)} \rightarrow \widehat{O}_0 \quad \text{as } h \rightarrow 0.$$

Discretization of an isolated stochastic system

Consider the nonlinear SDE in \mathbb{R}^d with additive noise,

$$dX_t = f(X_t) dt + \alpha dW_t,$$

where the drift coefficient f is continuously differentiable and satisfies the one-sided dissipative Lipschitz condition with constant L .

The drift-implicit Euler-Maruyama scheme with constant step size $h > 0$ applied to this SDE is

$$X_{n+1} = X_n + hf(X_{n+1}) + \alpha \Delta W_n,$$

which is, in general, an implicit algebraic equation and must be solved numerically for X_{n+1} for each n .

The difference of any two solutions

$$X_{n+1} - X'_{n+1} = (X_n - X'_n) + h (f(X_{n+1}) - f(X'_{n+1})),$$

does not contain a driving noise term. Then

$$\begin{aligned} |X_{n+1} - X'_{n+1}|^2 &= \langle X_{n+1} - X'_{n+1}, X_n - X'_n \rangle \\ &\quad + h \langle X_{n+1} - X'_{n+1}, f(X_{n+1}) - f(X'_{n+1}) \rangle \\ &\leq |X_{n+1} - X'_{n+1}| |X_n - X'_n| - hL |X_{n+1} - X'_{n+1}|^2, \end{aligned}$$

$$\Rightarrow |X_{n+1} - X'_{n+1}| \leq \frac{1}{1 + Lh} |X_n - X'_n|,$$

$$\Rightarrow |X_n - X'_n| \leq \frac{1}{(1 + Lh)^n} |X_0 - X'_0| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

i.e. all numerical solutions converge pathwise to each other forward in time.

Change variables to $U_n := X_n - \widehat{O}_n^{(h)}$, where $\widehat{O}_n^{(h)}$ is the numerical Ornstein-Uhlenbeck process, to obtain the numerical scheme

$$U_{n+1} = U_n + hf \left(U_{n+1} + \widehat{O}_{n+1}^{(h)} \right) + h\widehat{O}_n^{(h)}.$$

Taking the inner product of both sides with U_{n+1} we obtain

$$\begin{aligned} |U_{n+1}|^2 &= \langle U_{n+1}, U_n \rangle + h \left\langle U_{n+1}, f \left(U_{n+1} + \widehat{O}_{n+1}^{(h)} \right) \right\rangle + h \left\langle U_{n+1}, \widehat{O}_n^{(h)} \right\rangle \\ &\leq |U_{n+1}| |U_n| + h \left\langle U_{n+1}, f \left(U_{n+1} + \widehat{O}_{n+1}^{(h)} \right) \right\rangle + h |U_{n+1}| \left| \widehat{O}_n^{(h)} \right|. \end{aligned}$$

Rearranging, using the one-sided Lipschitz condition and simplifying gives

$$|U_{n+1}| \leq |U_n| - Lh |U_{n+1}| + h \left| f \left(\widehat{O}_{n+1}^{(h)} \right) \right| + h \left| \widehat{O}_n^{(h)} \right|.$$

$$\Rightarrow |U_{n+1}| \leq \frac{1}{1+Lh} |U_n| + \frac{h}{1+Lh} B_n^{(h)},$$

where

$$B_n(h) := \left| f\left(\widehat{O}_{n+1}^{(h)}\right) \right| + \left| \widehat{O}_n^{(h)} \right|,$$

$$\Rightarrow |U_n| \leq \frac{1}{(1+Lh)^{n-n_0}} |U_{n_0}| + \frac{h}{1+Lh} \sum_{j=n_0}^{n-1} \frac{1}{(1+h)^{n-1-j}} B_j^{(h)}.$$

Taking the pullback limit as $n_0 \rightarrow -\infty$ with n fixed, it follows that U_n is pathwise pullback absorbed into the ball $B_d[0, \bar{R}_n]$ in \mathbb{R}^d centered on the origin with squared radius

$$\bar{R}_n^2 := 1 + \frac{h}{1+Lh} \sum_{j=-\infty}^{n-1} \frac{1}{(1+h)^{n-1-j}} B_j^{(h)}.$$

Note that \bar{R}_n is random, but finite.

From the theory of random dynamical systems we conclude that the discrete time random dynamical system generated by drift-implicit Euler-Maruyama scheme has a random attractor with component sets in the corresponding balls $B_d[0, \bar{R}_n]$.

Since all of the trajectories converge together pathwise forward in time, the random attractor consists of a single stochastic stationary process which we shall denote by $\hat{U}_n^{(h)}$.

Transforming back to the original variable, we have shown that the drift-implicit Euler-Maruyama scheme applied to the nonlinear SDE has a stochastic stationary solution

$$\hat{X}_n^{(h)} := \hat{U}_n^{(h)} + \hat{O}_n^{(h)}, \quad n \in \mathbb{Z},$$

taking values in the random balls $B_d[\hat{O}_n^{(h)}, \bar{R}_n]$, which attracts all other solutions pathwise in both the forward and pullback senses.

Discretization of the coupled stochastic systems

Consider the coupled stochastic system in \mathbb{R}^{2d} (now α, β are nonzero scalars)

$$dX_t = (f(X_t) + \nu(Y_t - X_t)) dt + \alpha dW_t^1,$$

$$dY_t = (g(Y_t) + \nu(X_t - Y_t)) dt + \beta dW_t^2,$$

The corresponding drift-implicit Euler-Maruyama scheme with constant step size,

$$X_{n+1} = X_n + h (f(X_{n+1}) + \nu(Y_{n+1} - X_{n+1})) + \alpha \Delta W_n^1,$$

$$Y_{n+1} = Y_n + h (g(Y_{n+1}) + \nu(X_{n+1} - Y_{n+1})) + \beta \Delta W_n^2,$$

can be written as the $2d$ -dimensional vector system

$$\mathfrak{X}_{n+1} = \mathfrak{X}_n + h (\mathbf{F}(\mathfrak{X}_{n+1}) + \nu B \mathfrak{X}_{n+1}) + A \Delta \mathbf{W}_n$$

with the $2d$ -dimensional vectors

$$\mathfrak{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{W}_t = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}, \quad \mathbf{F}(\mathfrak{x}) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix},$$

and the $2d \times 2d$ -matrices

$$A = \begin{bmatrix} \alpha I_d & 0 \\ 0 & \beta I_d \end{bmatrix}, \quad B = \begin{bmatrix} -I_d & I_d \\ I_d & -I_d \end{bmatrix},$$

where I_d is the $d \times d$ identity matrix.

The function $\mathbf{G} := \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ defined by

$$\mathbf{G}(\mathfrak{x}) := \mathbf{F}(\mathfrak{x}) + \nu B\mathfrak{x}$$

satisfies a dissipative one-sided Lipschitz condition with constant L .

i.e. the vector scheme has essentially the same structure as the scheme for the uncoupled nonlinear equation, but in a higher dimensional space.

The previous analysis can be repeated almost verbatim to give the existence of a unique stochastic stationary process

$$\widehat{\mathfrak{X}}_n^{(h,\nu)} = \begin{pmatrix} \widehat{X}_n^{(h,\nu)} \\ \widehat{Y}_n^{(h,\nu)} \end{pmatrix}, \quad n \in \mathbb{Z},$$

which attracts all other solutions pathwise in both the forward and pull-back senses.

Moreover, the $\widehat{\mathfrak{X}}_n^{(h,\nu)}$ take values in the random balls $B_{2d}[\widehat{\mathfrak{D}}_n^{(h)}, \widehat{R}_n]$ for appropriately defined \widehat{R}_n (which are independent of ν), where $\widehat{\mathfrak{D}}_n^{(h)}$ is the discrete time Ornstein-Uhlenbeck stochastic stationary solution for the discrete time $2d$ -dimensional linear system

$$\mathfrak{X}_{n+1} = \frac{1}{1+h} \mathfrak{X}_n + \frac{1}{1+h} A \Delta \mathbf{W}_n.$$

Theorem 1

$$\begin{pmatrix} \widehat{X}_n^{(h,\nu)} \\ \widehat{Y}_n^{(h,\nu)} \end{pmatrix} \rightarrow \begin{pmatrix} \widehat{Z}_n^{(h,\infty)} \\ \widehat{Z}_n^{(h,\infty)} \end{pmatrix}$$

pathwise uniformly on bounded integer intervals $[N_1, N_2]$ as $\nu \rightarrow \infty$, where $(\widehat{Z}_n^{(h,\infty)})_{n \in \mathbb{Z}}$ is the discrete time stationary stochastic solution of the drift-implicit Euler-Maruyama scheme with constant step size

$$Z_{n+1} = Z_n + \frac{1}{2}h (f(Z_{n+1}) + g(Z_{n+1})) + \frac{1}{2}\alpha \Delta W_n^1 + \frac{1}{2}\beta \Delta W_n^2$$

applied to the averaged SDE

$$dZ_t = \frac{1}{2} (f(Z_t) + g(Z_t)) dt + \frac{1}{2}\alpha dW_t^1 + \frac{1}{2}\beta dW_t^2.$$

Synchronization of SDE with linear noise

A Stratonovich stochastic differential equation with linear noise

$$dX_t = f(X_t) dt + \alpha X_t \circ dW_t$$

can be transformed to the pathwise random ordinary differential equation

$$\frac{dx}{dt} = F(x, \bar{O}_t(\omega)) := e^{-\bar{O}_t(\omega)} f\left(e^{\bar{O}_t(\omega)} x\right) + \bar{O}_t(\omega) x$$

using the transformation

$$x(t, \omega) = e^{-\bar{O}_t(\omega)} X_t(\omega).$$

with the Ornstein-Uhlenbeck process $\bar{O}_t := \alpha e^{-t} \int_{-\infty}^t e^s dW_s$.

NOTE: F satisfy the one-sided Lipschitz condition if f does.

Similar a pair of Stratonovich SDEs

$$dX_t = f(X_t) dt + \alpha X_t \circ dW_t^1,$$

$$dY_t = g(Y_t) dt + \beta Y_t \circ dW_t^2,$$

can be transformed to the RODEs

$$\frac{dx}{dt} = F(x, \bar{O}_t^1(\omega)) := e^{-\bar{O}_t^1(\omega)} f\left(e^{\bar{O}_t^1(\omega)} x\right) + \bar{O}_t^1(\omega) x,$$

$$\frac{dy}{dt} = G(y, \bar{O}_t^2(\omega)) := e^{-\bar{O}_t^2(\omega)} g\left(e^{\bar{O}_t^2(\omega)} y\right) + \bar{O}_t^2(\omega) y,$$

with the transformations

$$x(t, \omega) = e^{-\bar{O}_t^1(\omega)} X_t(\omega) \qquad \bar{O}_t^1 := \alpha e^{-t} \int_{-\infty}^t e^s dW_s^1$$

$$y(t, \omega) = e^{-\bar{O}_t^2(\omega)} Y_t(\omega). \qquad \bar{O}_t^2 := \beta e^{-t} \int_{-\infty}^t e^s dW_s^2$$

The coupled system of random ordinary differential equations (RODEs)

$$\frac{dx}{dt} = F(x, \bar{O}_t^1(\omega)) + \nu(y - x),$$

$$\frac{dy}{dt} = G(y, \bar{O}_t^2(\omega)) + \nu(x - y)$$

has a pathwise asymptotically stable random attractor consisting of single stochastic stationary process $(\bar{x}_\nu(t, \omega), \bar{y}_\nu(t, \omega))$ with

$$(\bar{x}_\nu(t, \omega), \bar{y}_\nu(t, \omega)) \rightarrow (\bar{z}(t, \omega), \bar{z}(t, \omega)) \quad \text{as } \nu \rightarrow \infty,$$

where $\bar{z}(t, \omega)$ is the pathwise asymptotically stable stochastic stationary process of the averaged RODE

$$\frac{dz}{dt} = \frac{1}{2} [F(z, \bar{O}_t^1) + G(z, \bar{O}_t^2)]$$

i.e.

$$\frac{dz}{dt} = \frac{1}{2} \left[e^{-\bar{O}_t^1} f(\bar{O}_t^1 z) + e^{-\bar{O}_t^2} f(\bar{O}_t^2 z) \right] + \frac{1}{2} [\bar{O}_t^1 + \bar{O}_t^2] z$$

or the equivalent Stratonovich SDE

$$dZ_t = \frac{1}{2} \left[e^{-\eta_t} f(e^{\eta_t} Z_t) + e^{\eta_t} g(e^{-\eta_t} Z_t) \right] dt + \frac{1}{2} \alpha \circ dW_t^1 + \frac{1}{2} \beta \circ dW_t^2.$$

where $\eta_t := \frac{1}{2}(\bar{O}_t^1 - \bar{O}_t^2)$.

Direct synchronization of the SDE

The corresponding system of coupled SDE is

$$\begin{aligned}dX_t &= \left(f(X_t) + \nu \left(e^{\bar{O}_t^1 - \bar{O}_t^2} Y_t - X_t \right) \right) dt + \alpha X_t \circ dW_t^1, \\dY_t &= \left(g(Y_t) + \nu \left(e^{-\bar{O}_t^1 + \bar{O}_t^2} X_t - Y_t \right) \right) dt + \beta Y_t \circ dW_t^2.\end{aligned}$$

has a unique stochastic stationary solution

$$(\bar{X}_t^\nu(\omega), \bar{Y}_t^\nu(\omega))$$

which is pathwise globally asymptotically stable with

$$(\bar{X}_t^\nu(\omega), \bar{Y}_t^\nu(\omega)) \rightarrow \left(\bar{z}(t, \omega) e^{-\bar{O}_t^1(\omega)}, \bar{z}(t, \omega) e^{-\bar{O}_t^2(\omega)} \right) \quad \text{as } \nu \rightarrow \infty,$$

pathwise on finite time intervals $[T_1, T_2]$ of \mathbb{R}

Stochastic reaction-diffusion system on a thin two-layer domain

Let $D_{1,\varepsilon}$ and $D_{2,\varepsilon}$ be thin bounded domains in \mathbb{R}^{d+1} , $d \geq 1$,

$$D_{1,\varepsilon} = \Gamma \times (0, \varepsilon), \quad D_{2,\varepsilon} = \Gamma \times (-\varepsilon, 0),$$

with $0 < \varepsilon \leq 1$ and Γ a bounded C^2 -domain in \mathbb{R}^d .

Write

$$x \in D_\varepsilon := D_{1,\varepsilon} \cup D_{2,\varepsilon} \quad \text{as} \quad x = (x', x_{d+1})$$

where

$$x' \in \Gamma \quad \text{and} \quad x_{d+1} \in (-\varepsilon, 0) \cup (0, \varepsilon).$$

Consider the system of Ito stochastic PDE

$$\frac{\partial}{\partial t} U^i - \nu_i \Delta U^i + aU^i + f_i(U^i) + h_i(x) = \dot{W}(t, x'),$$
$$t > 0, \quad x \in D_{i,\varepsilon}, \quad i = 1, 2,$$

where $\dot{W}(t, x')$ white noise depending only $x' \in \Gamma$.

[Deterministic model: *Chueshov & Rekalov* (Matem. Sbornik, 2004)]

[Stochastic model: *Caraballo, Kloeden & Chueshov* (SIAM J. Math. Anal., 2007)]

Neumann boundary conditions

$$(\nabla U^i, n_i) = 0, \quad x \in \partial D_{i,\varepsilon} \setminus \Gamma, \quad i = 1, 2,$$

on the external part of the boundary of the compound domain D_ε , where n is the outer normal to ∂D_ε

Matching condition on Γ

$$\left(-\nu_1 \frac{\partial U^1}{\partial x_{d+1}} + k(x', \varepsilon)(U^1 - U^2) \right) \Big|_\Gamma = 0,$$

$$\left(\nu_2 \frac{\partial U^2}{\partial x_{d+1}} + k(x', \varepsilon)(U^2 - U^1) \right) \Big|_\Gamma = 0.$$

Synchronization as $\varepsilon^{-1} \rightarrow \infty$ with the averaged system

$$\frac{\partial}{\partial t}U - \nu\Delta_{\Gamma}U + aU + f(U) + h(x') = \dot{W}(t, x'), \quad x' \in \Gamma,$$

on the spatial domain Γ with the Neumann boundary conditions on $\partial\Gamma$ and with

$$\nu = \frac{\nu_1 + \nu_2}{2},$$

$$f(U) = \frac{f_1(U) + f_2(U)}{2}, \quad h(x') = \frac{h_1(x', 0) + h_2(x', 0)}{2}.$$

Method : Transform Ito SPDE into a pathwise random PDE.

Assumptions

- $f_i \in C^1(\mathbb{R})$ such that $f'_i(v) \geq -c$ for all $v \in \mathbb{R}$ and

$$vf_i(v) \geq a_0|v|^{p+1} - c, \quad |f'_i(v)| \leq a_1|v|^{p-1} + c,$$

a_j and c positive constants and $1 \leq p < 3$;

- $h_i \in H^1(D_{i,1})$, $i = 1, 2$;

- $k(\cdot, \varepsilon) \in L^\infty(\Gamma)$, $k(x', \varepsilon) > 0$, $x' \in \Gamma$, $\varepsilon \in (0, 1]$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} k(x', \varepsilon) = +\infty, \quad x' \in \Gamma;$$

- $W(t)$, $t \in \mathbb{R}$, two-sided $L_2(\Gamma)$ -valued Wiener with covariance operator $K = K^* \geq 0$ such that for some $\beta > \max\left\{1, \frac{d}{4}\right\}$

$$\text{tr} \left[K (-\Delta_N + 1)^{2\beta-1} \right] < \infty,$$

Δ_N Laplacian in $L_2(\Gamma)$ with Neumann boundary conditions on $\partial\Gamma$.

- $(\Omega, \mathcal{F}, \mathbb{P})$ the corresponding probability space

Theorem 2 *Under the above Assumptions the following assertions hold.*

1. *The coupled SPDE generates an RDS $(\theta, \bar{\phi}_\epsilon)$ in the space*

$$\mathcal{H}_\epsilon = L_2(D_{1,\epsilon}) \oplus L_2(D_{2,\epsilon}) \sim L_2(D_\epsilon)$$

given by $\bar{\phi}_\epsilon(t, \omega)U_0 = U(t, \omega)$, where $U(t, \omega) = (U^1(t, \omega); U^2(t, \omega))$ is a strong solution to the problem and $U_0 = (U_0^1; U_0^2)$.

2. *Similarly, the averaged SPDE generates an RDS $(\theta, \bar{\phi}_0)$ in the space $L_2(\Gamma)$.*

3. *Cocycles $\bar{\phi}_\epsilon$ converge to $\bar{\phi}_0$ in the sense*

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [0, T]} \frac{1}{\epsilon} \int_{D_\epsilon} |\bar{\phi}_\epsilon(t, \omega)v - \bar{\phi}_0(t, \omega)v|^2 dx = 0, \quad \forall \omega,$$

for any $v(x) \in \mathcal{H}_\epsilon$ independent of the variable x_{d+1} , and any $T > 0$.

4. *These RDS $(\theta, \bar{\phi}_\epsilon)$ and $(\theta, \bar{\phi}_0)$ have random compact pullback attractors $\{\bar{\mathfrak{A}}^\epsilon(\omega)\}$ and $\{\bar{\mathfrak{A}}^0(\omega)\}$ in their corresponding state spaces. Moreover, if K is non-degenerate, then $\{\bar{\mathfrak{A}}^0(\omega)\}$ is a singleton, i.e. $\bar{\mathfrak{A}}^0(\omega) = \{\bar{v}_0(\omega)\}$, where $\bar{v}_0(\omega)$ is an $L_2(\Gamma)$ -valued tempered random variable.*

5. The attractors $\{\bar{\mathfrak{A}}^\epsilon(\omega)\}$ are upper semi-continuous as $\epsilon \rightarrow 0$ in the sense that for all $\omega \in \Omega$

$$\lim_{\epsilon \rightarrow 0} \sup_{v \in \bar{\mathfrak{A}}^\epsilon(\omega)} \left\{ \inf_{v_0 \in \bar{\mathfrak{A}}^0(\omega)} \frac{1}{\epsilon} \int_{D_\epsilon} |v(x', x_{d+1}) - v_0(x')|^2 dx \right\} = 0.$$

6. In addition, if

$$\begin{aligned} \nu_1 = \nu_2 &:= \nu, & f_1(U) = f_2(U) &:= f(U), \\ h_1(x', x_{d+1}) &= h(x') = h_2(x', x_{d+1}); \end{aligned}$$

$f(U)$ is globally Lipschitz, i.e. there exists a constant $L > 0$ such that

$$|f(U) - f(V)| \leq L|U - V|, \quad U, V \in \mathbb{R},$$

and also that

$$k(x', \epsilon) > k_\epsilon \text{ for } x' \in \Gamma, \epsilon \in (0, 1] \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \epsilon^{-1} k_\epsilon = +\infty,$$

then, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ the random pullback attractor $\{\bar{\mathfrak{A}}^\epsilon(\omega)\}$ for $(\theta, \bar{\phi}_\epsilon)$ has the form

$$\bar{\mathfrak{A}}^\epsilon(\omega) \equiv \{v(x', x_{d+1}) \equiv v_0(x') : v_0 \in \bar{\mathfrak{A}}^0(\omega)\},$$

where $\{\bar{\mathfrak{A}}^0(\omega)\}$ is the random pullback attractor for the RDS $(\theta, \bar{\phi}_0)$.

References

L. Arnold, *Random Dynamical Systems*, Springer-Verlag, 1998.

A.N. Carvalho, H.M. Rodrigues & T. Dlotko, Upper semicontinuity of attractors and synchronization, *J. Math. Anal. Applns.*, 220 (1998), 13–41.

H.M. Rodrigues, Abstract methods for synchronization and applications, *Applicable Anal.*, 62 (1996), 263–296.

P.E. Kloeden, Synchronization of nonautonomous dynamical systems, *Elect. DJ. Diff. Eqns.*, 2003, 1-10.

T. Caraballo & P.E. Kloeden, The persistence of synchronization under environmental noise, *Proc. Roy. Soc. London A*461 (2005), 2257–2267.

T. Caraballo, P.E. Kloeden and A. Neuenkirch, Synchronization of systems with multiplicative noise, *Stochastics & Dynamics* 8 (2008), 139–154.

P.E. Kloeden, A. Neuenkirch and R. Pavani, Synchronization of noisy dissipative systems under discretization, *J. Difference Eqns. Applns.* (to appear)

T. Caraballo, I. Chueshov & P.E. Kloeden, Synchronization of a stochastic reaction-diffusion system on a thin two-layer domain, *SIAM J. Math. Anal.*, 38 (2007), 1489-1507.

I.D. Chueshov and A.M. Rekaló, Global attractor of contact parabolic problem on thin two-layer domain, *Sbornik Mathematics*, 195 (2004), 103–128