New Estimates for Invariance Entropy

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Outline

The concept of invariance entropy

Setting

Consider a continuous-time control system

$$\dot{x}(t) = F(x(t), u(t)), \quad u \in \mathcal{U},$$

on a Riemannian manifold M such that $F : M \times \mathbb{R}^m \to TM$ is continuous and continuously differentiable in the first argument. Then there are unique solutions $\varphi(t, x, u)$ for all $x \in M$ and $u \in \mathcal{U}$ and $\varphi_{t,u}(x) = \varphi(t, x, u)$ is continuously differentiable. Let $Q \subset M$ be a compact and controlled invariant set:

$$\forall x \in Q : \ \exists u \in \mathcal{U} : \ \varphi(t, x, u) \in Q \text{ for all } t \geq 0.$$

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Question:

How fast does the number of open-loop control functions, which are needed to stay in Q up to time τ, grow when τ goes to infinity?

The concept of invariance entropy

Definition (Fritz)

A set $\mathcal{S} \subset \mathcal{U}$ is called (τ, Q)-spanning if

$$\forall x \in Q: \ \exists u \in \mathcal{S}: \ \varphi(t, x, u) \in Q \text{ for all } t \in [0, \tau].$$

Let $r_{\text{inv}}(\tau, Q)$ denote the minimal cardinality of such a set and define the (strict) invariance entropy by

$$h_{\text{inv}}(Q) := \lim_{\tau \to \infty} \frac{1}{\tau} \ln r_{\text{inv}}(\tau, Q).$$

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5. $h_{\text{inv}}(Q)$ equals the infimal data rate in a feedback loop necessary to render the set Q invariant by a causal coding and control law.

Example I

Consider a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u \in \mathcal{U},$$

with compact control range. Assume that Q is a compact controlled invariant set with positive Lebesgue measure. Then

$$h_{\mathrm{inv}}(Q) = \sum_{\lambda \in \mathrm{spec}(A)} \max\{0, \mathrm{Re}\,\lambda\},$$

where every eigenvalue is counted with its multiplicity.

Example II

Consider a control-affine system of the form

$$\dot{x} = f(x) + u(t)g(x), \quad u \in \mathcal{U},$$

on $\mathbb{R}$ with $u(t) \in [a, b]$, $a < b$. Let D be a bounded control set with nonvoid interior and assume that the system is locally accessible on $\operatorname{cl} D$. Then for $Q = \operatorname{cl} D$ and $K \subset D$

$$\begin{aligned} h_{\mathrm{inv}}(K, Q) &= \max\{0, \inf \Sigma_{\mathrm{Ly}}(Q)\} \\ &= \max\left\{0, \min_{x \in Q}\left[f'(x) - \frac{f(x)}{g(x)} g'(x)\right]\right\}. \end{aligned}$$

Lower Bounds: the basic idea

Let m be an outer measure on M such that $0 < m(Q) < \infty$. Define

$$Q(u,\tau) := \{x \in Q \ : \ \varphi([0,\tau],x,u) \subset Q\}, \quad u \in \mathcal{U}, \ \tau > 0.$$

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Let $\mathcal{S} \subset \mathcal{U}$ be a minimal (τ, Q)-spanning set. Then

$$Q = \bigcup_{u \in \mathcal{S}} Q(u,\tau) \quad \Rightarrow \quad m(Q) \leq \sum_{u \in \mathcal{S}} m(Q(u,\tau)).$$

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This implies

$$m(Q) \leq \#\mathcal{S} \cdot \sup_u m(Q(u,\tau)) = r_{\text{inv}}(\tau, Q) \cdot \sup_u m(Q(u,\tau)).$$

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Therefore,

$$\boxed{h_{\mathrm{inv}}(Q) \geq -\limsup_{\tau \to \infty} \frac{1}{\tau} \ln \sup_u m(Q(u,\tau)).}$$

Reduction of the problem

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Idea (from the theory of escape rates):

Introduce the Bowen-metrics on M:

$$d_{u,\tau}(x,y) := \max_{t\in[0,\tau]} d(\varphi(t,x,u),\varphi(t,y,u)).$$

Cover the set $Q(u,\tau)$ with a minimal collection of Bowen-balls:

$$Q(u,\tau) \subset \bigcup_{x\in S_{u,\tau,\varepsilon}} B_\varepsilon^{u,\tau}(x), \quad \varepsilon > 0 \text{ (fixed)}$$

This implies

$$m(Q(u,\tau)) \leq \sum_x m(B_\varepsilon^{u,\tau}(x)) \leq \#S_{u,\tau,\varepsilon} \cdot \sup_{x\in Q(u,\tau)} m(B_\varepsilon^{u,\tau}(x)).$$

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Katrin Gelfert's Lemma (simplified version):

Consider a dynamical system of class C^1 on a Riemannian manifold:

$$\varphi : \mathbb{T} \times M \to M, \quad (t,p) \mapsto \varphi^t(p).$$

Let K be a compact set and $E \subset T_K M$ a subbundle such that

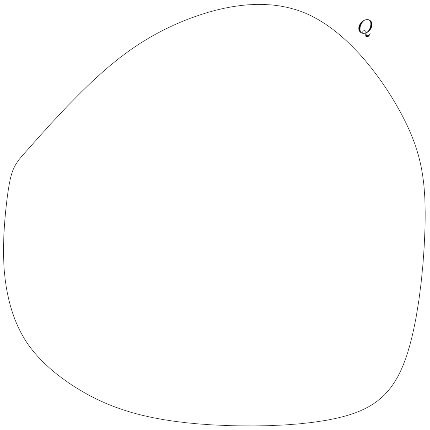
$$\inf_{p \in K} \left| \det d_p \varphi^t|_{E_p} \right| > 1 \text{ for some } t > 0.$$

Then there is $\tilde{\varepsilon}(t) > 0$ such that for all $p \in K$ and $\varepsilon \in (0, \tilde{\varepsilon}]$:

$$\mu_H(B_\varepsilon^t(p), \dim M, \varepsilon) \leq \mathrm{const} \cdot \varepsilon^{\dim M} \left| \det d_p \varphi^t|_{E_p} \right|^{-1},$$

where $\mu_H(\cdot, \dim M, \varepsilon)$ denotes outer Hausdorff measure.

Summary



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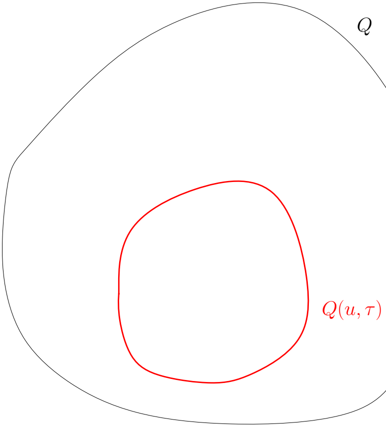

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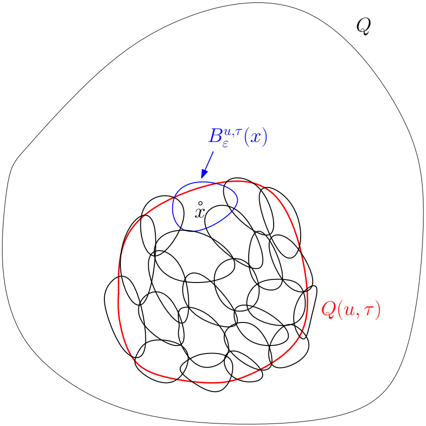
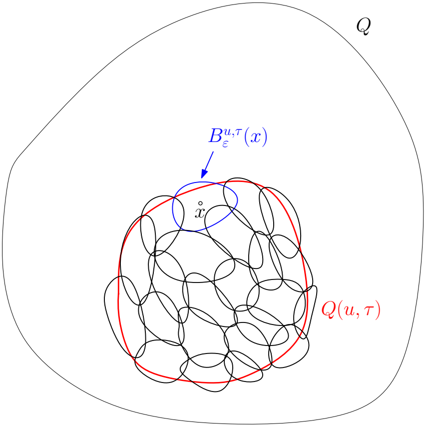


SUMMARY



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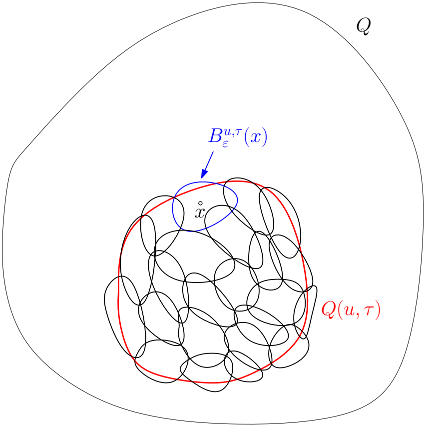

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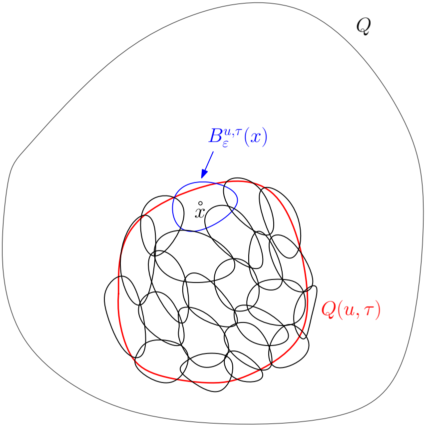

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$$m(B_\varepsilon^{u,\tau}(x)) \leq \frac{C\varepsilon^d}{\left|\det d_x\varphi_{t,u}|_{E_{u,x}}\right|}$$

for $m = \mu_H(\cdot, \dim M, \varepsilon)$.

(Nonautonomous version of Katrin Gelfert's Lemma)

The main result

Definition

Define

$$\mathcal{Q} := \left\{(u,x) \in \mathcal{U} \times M \ : \ \varphi(\mathbb{R}_0^+, x, u) \subset Q\right\}.$$

$$h_{\mathrm{esc}}(Q) := \limsup_{\tau \to \infty} \frac{1}{\tau} \ln \left[\limsup_{\varepsilon \searrow 0} \sup_{u \in \pi_{\mathcal{U}} \mathcal{Q}} \varepsilon^{\dim M} \# S_{u,\tau,\varepsilon}\right].$$

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Theorem

Let $E \to \mathcal{Q}$ be a subbundle of the vector bundle

$$\bigcup_{(u,x)\in\mathcal{Q}} \{u\} \times T_x M \to \mathcal{Q}, \quad (u, v \in T_x M) \mapsto (u,x),$$

with $\inf_{x:\ (u,x)\in\mathcal{Q}} \left|\det d_x\varphi_{\tau,u}|_{E_{u,x}}\right| > 1$ for all $\tau \geq \tau_0$ and $u \in \pi_{\mathcal{U}}\mathcal{Q}$. Then

$$h_{\mathrm{inv}}(Q) \geq \limsup_{\tau\to\infty} \frac{1}{\tau} \inf_{(u,x)\in\mathcal{Q}} \ln\left|\det d_x\varphi_{\tau,u}|_{E_{u,x}}\right| - h_{\mathrm{esc}}(Q).$$

Generalized liouville formula

Proposition

Assume that the subbundle E in the theorem is equivariant. Then

$$\ln \left| \det d_x \varphi_{\tau,u}|_{E_{u,x}} \right| = \int_0^\tau \underbrace{\mathrm{tr}\left[\nabla F_{u(s)}(\varphi_{s,u}(x)) \circ Q(\Theta_s u, \varphi_{s,u}(x))\right]}_{\text{partial divergence of } F_{u(s)}} ds,$$

where $Q(u,x) : T_x M \to E_{u,x}$ is the orthogonal projection.

The "escape entropy" $h_{\text{esc}}(Q)$

Under mild assumptions, $|h_{\text{esc}}(Q)| < \infty$. In some cases we can show that

$$h_{\text{esc}}(Q) \leq 0,$$

and hence we can omit it in the estimate for invariance entropy:

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1. Uniformly expanding systems: On Q it holds that

$$d(\varphi(t,x,u),\varphi(t,y,u)) \geq ce^{\lambda t} d(x,y), \quad t \geq 0 \quad (c,\lambda > 0).$$

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2. Inhomogeneous bilinear systems (under mild conditions):

$$\dot{x} = \left[A_0 + \sum_{i=1}^{m} u_i(t) A_i\right] x + Bv(t), \quad (u,v) \in \mathcal{U} \times \mathcal{V}.$$

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3. (?) Bilinear systems on flag manifolds $\mathbb{F}(d_1,\ldots,d_k)$, $Q =$ closure of a control set (joint work with Luiz San Martin).

Remark

From our theorem we can recover the formula for linear systems: If

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u \in \mathcal{U},$$

we can define an equivariant subbundle $E \to \mathcal{Q}$ by setting

$$E_{u,x} :\equiv \mathbb{E}^+(A) \quad \text{(the unstable subspace)}$$

Applying our theorem we obtain

$$h_{\mathrm{inv}}(Q) \geq \limsup_{\tau \to \infty} \frac{1}{\tau} \inf_{(u,x) \in \mathcal{Q}} \int_0^\tau \operatorname{tr} A|_{\mathbb{E}^+(A)} \, ds - h_{\mathrm{esc}}(Q).$$

Since $h_{\mathrm{esc}}(Q) \leq 0$ in this case, we obtain

$$h_{\mathrm{inv}}(Q) \geq \operatorname{tr} A|_{\mathbb{E}^+(A)} = \sum_{\lambda \in \mathrm{spec}(A)} \max \{0, \operatorname{Re} \lambda\}.$$

Happy Birthday, Fritz!