LOCAL CONTROLLABILITY AND THE ATKINSON SPECTRAL PROBLEM

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What is an Atkinson spectral problem?

Let I_n be the $n \times n$ unit matrix. Let $J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$ be the standard $2n \times 2n$ dimensional antisymmetric matrix. Let $H(\cdot)$, $\Gamma(\cdot)$ be two bounded continuous functions which take values in the set $\mathbb{S}_{2n}$ of symmetric $2n \times 2n$ matrices. Suppose that $\Gamma(t) \geq 0$ for all $t \in \mathbb{R}$.

Consider the boundary value problem

$$J\frac{dz}{dt} = (H(t) + \lambda\Gamma(t))z \qquad z \in \mathbb{C}^{2n} \qquad (1)$$
$$z(-\infty) = z(\infty) = 0$$

where $\lambda \in \mathbb{C}$ and $-\infty < t < \infty$.

One can impose boundary conditions at the endpoints of a finite interval $[-a, a]$; one obtains problem (1) by "letting $a \to \infty''$. There is a spectral theory for the problem (1) which is analogous in many respects to that of a self-adjoint operator on a Hilbert space. We indicate the bare-bones elements of this spectral theory. Let $\Phi(t)$ be the fundamental matrix solution of $Jz' = H(t)z$.

- One imposes an <u>Atkinson Condition</u>, of the form

$$\int_{-\infty}^{+\infty} |\Gamma(t)\Phi(t)z_0|^2 \, dt > 0 \text{ if } 0 \neq z_0 \in \mathbb{R}^{2n}$$

- One introduces the Weyl m-functions $m_{\pm}(\lambda)$ $(Im\lambda \neq 0)$. They assume values in the set $\mathbb{S}_n(\mathbb{C})$ of symmetric complex $n \times n$ matrices. One defines the "characteristic function" $g(\lambda)$ $(Im\lambda \neq 0)$ in terms of $m_{\pm}(\lambda)$; it assumes values in $\mathbb{S}_{2n}(\mathbb{C})$.

- One proves the existence of a spectral matrix $\rho(\cdot) \in \mathbb{S}_{2n}$ with the property

$$Im\, g(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\rho(dt)}{|t-\lambda|^2}$$

  The spectrum of (1) is the set of increase points of $\rho(\cdot)$.

Example Consider the Schrödinger equation

$$-x'' + q(t)x = \lambda x \qquad x \in \mathbb{R}.$$

Put it in the form (1) by writing $z = \begin{pmatrix} x \\ x' \end{pmatrix}$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}' = \left[\begin{pmatrix} -q(t) & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ x' \end{pmatrix}$$
$$x(-\infty) = x(\infty) = 0$$

The Weyl m-functions $m_{\pm}(\lambda)$, the characteristic function $g(\lambda)$, and the spectral matrix for this problem are constructed in Coddington-Levinson, Chapter 9.

Let us now connect the Atkinson problem, especially the Atkinson condition, with linear nonautonomous control theory. Consider the control system

$$x' = A(t)x + B(t)u \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \tag{2}$$

where $A(\cdot)$, $B(\cdot)$ are bounded uniformly continuous matrix-valued functions of the appropriate dimensions. As is well known, this system is locally null controllable iff the controllability matrix is positive definite: that is, iff there exists $T > 0$, $\delta > 0$ such that

$$\int_0^T |B^t(s)\Psi^{-1}(s)^t x_0|^2 \, ds \geq \delta |x_0|^2 \ \forall x_0 \in \mathbb{R}^n.$$

Here $\Psi(t)$ is the fundamental matrix solution of $x' = A(t)x$.

We see that the Atkinson condition holds if $\exists T > 0$, $\delta > 0$ such that

$$\int_0^T |\Gamma(s)\Phi(s)z_0|^2 \geq \delta|z_0|^2 \quad z_0 \in \mathbb{R}^{2n}.$$

This is the local controllability condition for the adjoint system

$$z' = -H(t)Jz + \Gamma(t)u.$$

This connection turns out to be very useful in the context of the nonautonomous spectral problem (1), as we now indicate.

We treat the nonautonomous differential system (1) by introducing the Bebutov flow. Suppose that H and Γ are uniformly continuous and bounded. Let Ω be the Bebutov hull of (H, Γ) and let $\{\tau_t \mid t \in \mathbb{R}\}$ be the corresponding translation flow. There are continuous functions $\tilde{H}$, $\tilde{\Gamma} : \Omega \to \mathbb{S}_{2n}$ and a point $\omega_0 \in \Omega$ such that $H(t) = \tilde{H}(\tau_t(\omega_0))$, $\Gamma(t) = \tilde{\Gamma}(\tau_t(\omega_0))$. We abuse notation, and write H, Γ instead of $\tilde{H}$, $\tilde{\Gamma}$. We obtain a family of Atkinson problems

$$Jz' = [H(\tau_t(\omega)) + \lambda\Gamma(\tau_t(\omega))]z \qquad (1_\omega)$$

$$z(-\infty) = z(\infty) = 0$$

where ω ranges over Ω.

Say that a compact subset $M \subset \Omega$ is minimal if it is nonempty, $\{\tau_t\}$-invariant, and if the orbit $\{\tau_t(\omega) \mid t \in \mathbb{R}\}$ is dense in M for each $\omega \in M$.

Theorem (J.-Nerurkar, Colonius) Suppose that, for each minimal set $M \subset \Omega$, $\exists \omega_0 \in M$ such that the adjoint control system

$z' = -H(\tau_t(\omega_0))Jz + \Gamma(\tau_t(\omega_0))u$ is locally null controllable. Then the system is uniformly locally null controllable for all $\omega \in \Omega$. That is, $\exists T > 0$, $\delta > 0$ such that

$$\int_0^T |\Gamma(\tau_t(\omega))\Phi_\omega(t)z_0|^2 \, dt \geq \delta |z_0|^2 \quad z_0 \in \mathbb{R}^{2n}$$

for all $\omega \in \Omega$. So the Atkinson condition holds, uniformly in $\omega \in \Omega$.

In particular, if Ω is itself minimal, then one gets uniform local null controllablity from null controllability at a single point $\omega_0 \in \Omega$.

Here is another result concerning the nonautonomous Atkinson problem, which is useful in various contexts. First we recall a standard definition.

Definition The family (1_ω) is said to have an exponential dichotomy (ED) over Ω if there exist constants $K > 0$, $\gamma > 0$ and a projection-valued function $\omega \to P_\omega = P_\omega^2 \in \mathbb{M}_{2n}$ such that

$$\|\Phi_\omega(t) P_\omega \Phi_\omega(s)^{-1}\| \leq K e^{-\gamma(t-s)} \qquad t \geq s$$
$$\|\Phi_\omega(t)(I - P_\omega)\Phi_\omega(s)^{-1}\| \leq K e^{\gamma(t-s)} \qquad t \leq s.$$

Theorem (J.-Nerurkar; Fabbri-J.-Núñez)

Suppose that the Atkinson condition is valid, uniformly in $\omega \in \Omega$. Let $\lambda_1 < \lambda < \lambda_2$ be an open interval in $\mathbb{R}$. Suppose that the spectral matrrix $\rho_\omega(\cdot)$ is constant on (λ_1, λ_2) for each $\omega \in \Omega$. Then the family (1_ω) admits an ED over Ω.

Of course the proof uses the uniform Atkinson condition (alias the uniform local null controllablity of the family of adjoint systems) in a crucial way.

The Atkinson theory for nonautonomous differential systems (1) can be used to study various control-theoretic problems.

- Linear Regulator Problem (Feedback Control Problem)
- Linear Nonautonomous H^∞ Control
- Yakubovich Frequency Theorem
  - Absolute Stability
  - Dissipative Systems

In the rest of the talk, we will discuss the nonautonomous version of the Yakubovich theorem. This theorem was stated and applied by Yakubovich for periodic control systems. (Almost) all of his results carry over to nonautonomous control systems.

Consider a linear control system

$$x' = A(t)x + B(t)u \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \quad (2)$$
$$x(0) = x_0$$

together with a quadratic form

$$\mathcal{Q}(t,x,u) = \frac{1}{2}\{< x, G(t)x > +2 < x, g(t)u > + \\ + < u, R(t)u >\}$$

where A, B, G, g, R are bounded, uniformly continuous matrix-valued functions. It is assumed that $G(\cdot) \in \mathbb{S}_n$, $R(\cdot) \in \mathbb{S}_m$, and that $R(t) > 0$ for all $t \in \mathbb{R}$. It is <u>not</u> assumed that $\mathcal{Q}$ is positive semidefinite, indeed one might be interested in the case $g = 0$ and $G < 0$. It is assumed that the system (2) is uniformly controllable. This implies L^2-stabilizability of (2).

Let us introduce a Bebutov flow $(\Omega, \{\tau_t\})$ etc. and pass to families (2_ω) and $\mathcal{Q}_\omega(t, x, u)(\omega \in \Omega)$. For each $\omega \in \Omega$ consider the functional

$$\mathcal{I}_\omega = \int_0^\infty \mathcal{Q}_\omega(t, x(t), u(t))dt.$$

The problem is to determine $x(\cdot)$, $u(\cdot)$ which minimize $\mathcal{I}_\omega$, subject to (2_ω).

Yakubovich considered the case when the functions A, B, G, g, R are all T-periodic for some $T > 0$. He used the Pontryagin principle to convert the study of (2) and $\mathcal{Q}$ into the study of a periodic Hamiltonian system. Then he showed that the minimization problem is solvable if and only if the Hamiltonian system satisfies

- a Frequency Condition FR
- a Nonoscillation Condition NO

He also indicated a number of other important statements which are equivalent to FR + NO.

We indicate how his discussion can be generalized to the case when A, B, G, g, R are bounded and uniformly continuous. For each $\omega \in \Omega$ introduce the Hamiltonian

$$H_\omega =< y, x' > -\mathcal{Q}(t, x, u),$$

then write the Hamilton equatrions

$$x' = \frac{\partial H_\omega}{\partial y}$$
$$y' = -\frac{\partial H_\omega}{\partial x}.$$

By the Pontryagin Principle one has that, if $u \in L^2([0, \infty), \mathbb{R}^m)$ is a minimizing control, and if $x \in L^2([0, \infty), \mathbb{R}^n)$ is the corresponding solution of (2_ω), then

$$\frac{\partial H_\omega}{\partial u} = 0,$$

which implies the feedback rule

$$u = R^{-1}(B^t y - g^t x).$$

So it turns out that the Hamiltonian equations are

$$Jz' = \begin{pmatrix} -Q & A^t - gR^{-1}B^t \\ A - BR^{-1}g^t & BR^{-1}B^t \end{pmatrix} z \quad (3_\omega)$$

where all functions have argument $\tau_t(\omega)$. Here $Q = G - gR^{-1}g^t$ and $z = \begin{pmatrix} x \\ y \end{pmatrix}$. We write $\Phi(t)$ or $\Phi_\omega(t)$ for the fundamental matrix solution of (3_ω).

We introduce the conditions FR and NO which allow one to solve the minimization problem, and allow one to verify other results as well.

FR For each $\omega \in \Omega$, the equation (3_ω) does not admit a nontrivial bounded solution.

In the periodic case, the FR condition is equivalent to the condition that the period matrix $\Phi(T)$ of (3) does not admit an eigenvalue of modulus 1.

In the general nonautonomous case, the condition FR implies that the family (3_ω) admits an exponential dichotomy over Ω. This is a consequence of a well-known result of Sacker-Sell and Selgrade. For each $\omega \in \Omega$, let P_ω be the dichotomy projection, and let $\lambda_\omega = ImP_\omega \subset \mathbb{R}^{2n}$ be its image. It turns out that $dim\lambda_\omega$ is automatically equal to n. In fact, λ_ω is a Lagrange plane in the sense that, if $z_1, z_2 \in \lambda_\omega$, then $< z_1, Jz_2 >= 0$.

We now formulate the nonoscillation condition on a geometric way:

<u>**NO**</u> For each $\omega \in \Omega$, λ_ω does not contain a vector of the form $\begin{pmatrix} 0 \\ y \end{pmatrix}$ where $0 \neq y \in \mathbb{R}^n$. Thus λ_ω contains no "vertical" vector.

The NO condition can often be reformulated in terms of the Lidskii-Yakubovich angles. For example, let $\omega \in \Omega$, and write

$$\Phi_\omega(t) = \begin{pmatrix} U_1(t) & U_2(t) \\ V_1(t) & V_2(t) \end{pmatrix}.$$

Then set $Arg_3(t) = arg\, det(U_1(t) + iU_2(t))$ where the arg is determined in a continuous way. Then under mild conditions on Ω, the condition NO is implied by the condition that $Arg_3(t)$ be bounded for each $\omega \in \Omega$.

This can be verified by using the rotation number for linear Hamiltonian systems (J.-Nerurkar, Novo-Núñez-Obaya, Fabbri-J.-Núñez). One can often restate NO as follows: "the rotation number of the family (3_ω) equals zero".

Before stating the nonautonomous version of the Yakubovich theorem, we need to make two observations.

First, let $\lambda_\omega = ImP_\omega \subset \mathbb{R}^2 n$. Since λ_ω contains no vertical vector $\begin{pmatrix} 0 \\ y \end{pmatrix}$, there exists an $n \times n$ symmetric matrix $m_\omega \in \mathbb{S}_n$ such that

$$\lambda_\omega = Span \left\{ \begin{pmatrix} e_1 \\ m_\omega \cdot e_1 \end{pmatrix}, \dots \begin{pmatrix} e_n \\ m_\omega \cdot e_n \end{pmatrix} \right\}.$$

Without justifying the terminology, we call m_ω the Weyl m-matrix of the system (3_ω). If one fixes $\omega \in \Omega$ and introduces the map $m : t \to m_{\tau_t(\omega)} : \mathbb{R} \to \mathbb{S}_n$, then this map satisfies the Riccati equation

$$\frac{dm}{dt} + [A - gR^{-1}g^t]m + m[A - BR^{-1}g^t] + \\ + mBR^{-1}B^t m = G - gR^{-1}g^t \qquad (4_\omega)$$

where the functions A, B, G, g, R are evaluated along the orbit $\{\tau_t(\omega)\}$.

Second, suppose that, for some $\omega \in \Omega$, the minimization problem is solvable for each $x_0 \in \mathbb{R}^n$. The minimizing control u_0 determines uniquely an initial value y_0 of the dual variable y. It turns out that the set $\left\{ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \mid x_0 \in \mathbb{R}^n \right\} \subset \mathbb{R}^{2n}$ is a Lagrange plane $\tilde{\lambda}_\omega$.

Theorem (Fabbri-J.-Núñez) Suppose that the functions A, B, G, g, R are bounded and uniformly continuous, and that $R > 0$. Suppose that the control system (2_ω) is locally null controllable for each $\omega \in \Omega$. Then the following statements are equivalent.

(A) For each $\omega \in \Omega$, the problem of minimizing $\mathcal{I}_\omega$ subject to (2_ω) admits a solution for each $x_0 \in \mathbb{R}^n$. Let $\tilde{\lambda}_\omega$ be the corresponding Lagrange plane; then $\omega \to \tilde{\lambda}_\omega$ is continuous.

(B) The conditions FR and NO hold for the family of Hamiltonian systems (3_ω).

(C) There is a continuous function $\Omega \to \mathbb{S}_n$: $\omega \to m_\omega$ with the following properties.

First, for each $\omega \in \Omega$, the function $m : \mathbb{R} \to \mathbb{S}_n$ defined by $t \to m_{\tau_t(\omega)}$ satisfies the Riccati equation (4_ω). Second, if one sets $y = m_\omega x$, and then sets

$$u = R^{-1}(B^t y - g^t x) = R^{-1}(B^t m - g^t)x,$$

then the equation

$$x' = Ax + Bu = [A + BR^{-1}(B^t m - g^t)]x$$

is uniformly exponentially stable.

(D) There exist a positive number δ and a continuous function $m : \Omega \to \mathbb{S}_n$ such that, for each $\omega \in \Omega$, the Lyapunov function

$$V_\omega(t,x) = < x, m(\tau_t(\omega))x >$$

satisfies

$$\frac{dV_\omega}{dt} \leq 2\mathcal{Q}_\omega(t,x,u) - \delta(|x(t)|^2 + |u(t)|^2)$$

for each continuous function $u : [0, \infty) \to \mathbb{R}^m$. The derivative is calculated using equation (2_ω).

(E) The functional $\mathcal{I}_\omega$ is positive definite on the set of pairs $(x, u) \in L^2([0, \infty), \mathbb{R}^n) \times L^2([0, \infty), \mathbb{R}^m)$ which satisfy $x(0) = 0$. Thus $\exists \delta > 0$ such that for all such pairs and all $\omega \in \Omega$:

$$\int_0^\infty \mathcal{Q}_\omega(t, x(t), u(t))dt \geq \delta \int_0^\infty (|x(t)|^2 + |u(t)|^2)dt$$

It will come as no surprise that the function $\omega \to m_\omega$ encountered in parts (C) and (D) of the above theorem is just the Weyl function.

Let us now consider an application which is based on a recent paper of Yakubovich-Fradkov-Hill-Proskurnikov. That paper considers dissipative control systems and their storage functions.

We recall some definitions. Consider again the control system

$$x' = A(t)x + B(t)u. \qquad (2)$$

Let

$$\mathcal{Q}(t,x,u) = \frac{1}{2}\{< x, Gx > + < x, gu > + < u, Ru >\}$$

be a quadratic functional with $R > 0$.

The system (2) is said to be dissipative with supply rate $\mathcal{Q}$ if whenever $x(0) = 0$ and (x, u) satisfies (2), then

$$\int_0^{\infty} \mathcal{Q}(s, x(s), u(s))ds \geq 0 \text{ for all } t \geq 0.$$

It is strictly dissipative if there exists $\delta > 0$ such that, whenever $x(0) = 0$ and (x, u) satisfies (2), then

$$\int_0^{t} \mathcal{Q}(s, x(s), u(s))ds \geq \delta \int_0^{t} (|x(s)|^2 + |u(s)|^2)ds$$

$\forall t \geq 0$.

So if $\mathcal{Q}$ represents power, then the system (2) dissipates energy. Note that, in general, the quadratic form $\mathcal{Q}$ will take on negative as well as positive values.

Next, one says that the system (2) admits a <u>storage function</u> V with supply rate $\mathcal{Q}$ if $V(t,x) \geq 0$, $V(t,0) = 0$, and

$$\int_{t_1}^{t_2} \mathcal{Q}(s, x(s), u(s)) ds \geq V(t_2, x(t_2)) - V(t_1, x(t_1))$$

for all pairs (x, u) satisfying (2). The function V is called a <u>strong storage function</u> if in addition $V(t,x) > 0$ whenever $x \neq 0$.

Now suppose as before that A, B, G, g, R are all uniformly continuous and bounded, and introduce the Bebutov flow $(\Omega, \{\tau_t\})$ together with the corresponding functions A, B, G, g, R which are defined over Ω. Thus for each $\omega \in \Omega$, the control system (2_ω) has the supply rate $\mathcal{Q}_\omega$. Suppose that the conditions FR and NO are satisfied. Let m_ω be the Weyl matrix of the system (3_ω).

From part (E) of the previous theorem, we see that, for each $\omega \in \Omega$, the system (2_ω) is strictly dissipative with supply rate $\mathcal{Q}_\omega$.

From part (D) of the theorem, we see that, if $V_\omega(t,x) = < x, m_{\tau_t(\omega)} x >$ and if the Weyl matrix m_ω is positive definite for each $\omega \in \Omega$, then V_ω is a strong storage function for (2_ω) which corresponds to the supply rate $\mathcal{Q}_\omega$.

However this is not the end of the story. We can generalize a result of Y- F- H- P, as follows. Let P_ω be the dichotomy projection corresponding to the system (3_ω). Let $\lambda^-_\omega = Kern P_\omega = Im(I - P_\omega)$. When FR and NO are valid, it turns out that λ^-_ω contains no vertical vector $\begin{pmatrix} 0 \\ y \end{pmatrix}$, $0 \neq y \in \mathbb{R}^n$.

If $\omega \in \Omega$ we can write

$$\lambda_\omega = Span \left\{ \begin{pmatrix} e_1 \\ m_\omega^- \cdot e_1 \end{pmatrix}, \ldots, \begin{pmatrix} e_n \\ m_\omega^- \cdot e_n \end{pmatrix} \right\}$$

where m_ω^- is an $n \times n$ symmetric matrix. One can prove that, in the sense of symmetric matrices, there holds

$$m_\omega^- > m_\omega \qquad (\omega \in \Omega).$$

Fact If m_ω^- is positive definite for each $\omega \in \Omega$, then $V_\omega^-(t,x) = < x, m_\omega^- x >$ is a strong storage function for (2_ω) corresponding to the supply rate $\mathcal{Q}_\omega$.