From adaptive control to funnel control

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joint work over many years with
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Dedicated to Fritz on the occasion of his 60th birthday

System class: frequency domain

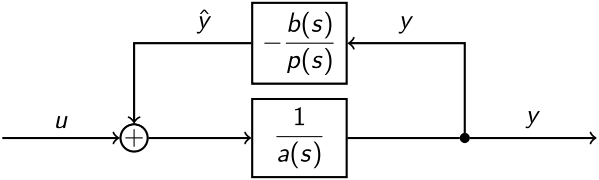
$$y(s) = \frac{p(s)}{q(s)}\, u(s) \qquad p, q \in \mathbb{R}[s] \text{ s.t.} \quad \mathrm{rdeg}\, \frac{p(s)}{q(s)} := \deg q - \deg p \geq 1,$$

Euclidean algorithm yields, for some $a, b \in \mathbb{R}[s]$

$$q(s) = a(s)\, p(s) + b(s), \qquad \deg b < \deg p, \qquad \deg a(s) = \mathrm{rdeg}\, \frac{p(s)}{q(s)}$$

and hence

$$y(s) = \tfrac{1}{a(s)} \left[-\tfrac{b(s)}{p(s)} y(s) + u(s) \right]$$



System class: time domain

$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\;u(t) \\ y(t) &= C\;x(t)\,, \end{aligned}}$$

$$A \in \mathbb{R}^{n\times n}$$
$$B, C^\top \in \mathbb{R}^{n\times m}$$

Then

$$G(s) \;=\; C(sI-A)^{-1}B \;=\; CB\,s^{-1} + CAB\,s^{-2} + \ldots + CA^{\rho-1}B\,s^{-\rho} + \ldots$$

We assume existence of a **strict relative degree**, i.e.

$$\rho = \mathrm{srdeg}\, G(s) = \sup\left\{ k\in\mathbb{Z} \;\middle|\; \lim_{s\to\infty} s^k\, G(s) \in \mathbf{Gl}_m(\mathbb{R}) \right\}$$

$$\Longleftrightarrow$$

$$CA^iB = 0 \quad \text{for } i = 0,\ldots,\rho-2 \quad \wedge \quad CA^{\rho-1}B \in \mathbf{Gl}_m(\mathbb{R})$$

Byrnes-Isidori form

$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}}$$

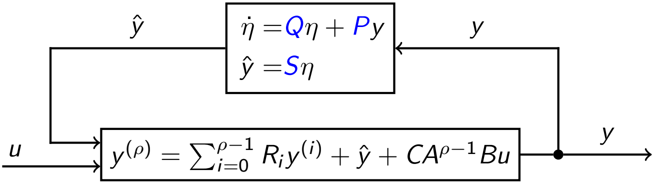
$$\operatorname{srdeg} C(sI - A)^{-1}B = \rho$$

$$\Longrightarrow \quad \exists\ T \in \mathbf{Gl}_n(\mathbb{R})\ :\ T\,x = \Big(y(t)^\top, \ldots, (y^{(\rho-1)}(t))^\top, \eta(t)^\top\Big)^\top$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} y(t) \\ \\ \vdots \\ \\ y^{(\rho-1)} \\ \hline \eta(t) \end{array}\right] = \left[\begin{array}{ccccc|c} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_0 & R_2 & \cdots & R_{\rho-2} & R_{\rho-1} & S \\ \hline P & 0 & \cdots & 0 & 0 & Q \end{array}\right] \left[\begin{array}{c} y(t) \\ \\ \vdots \\ \\ y^{(\rho-1)} \\ \hline \eta(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ CA^{\rho-1}B \\ \hline 0 \end{array}\right] u(t)$$

Byrnes-Isidori form: internal loop

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} y(t) \\ \vdots \\ y^{(\rho-1)} \\ \hline \eta(t) \end{array}\right] = \left[\begin{array}{cccccc|c} 0 & I_m & 0 & \cdots & 0 & & 0 \\ 0 & 0 & I_m & & & & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & I_m & & 0 \\ R_0 & R_2 & \cdots & R_{\rho-2} & R_{\rho-1} & & S \\ \hline P & 0 & \cdots & 0 & 0 & & Q \end{array}\right] \left[\begin{array}{c} y(t) \\ \vdots \\ y^{(\rho-1)} \\ \hline \eta(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ CA^{\rho-1}B \\ \hline 0 \end{array}\right] u(t)
$$



Minimum phase: frequency domain

$y(s) = \frac{p(s)}{q(s)}\, u(s)$ is **minimum phase** $:\Leftrightarrow$ $p(s)$ is Hurwitz.

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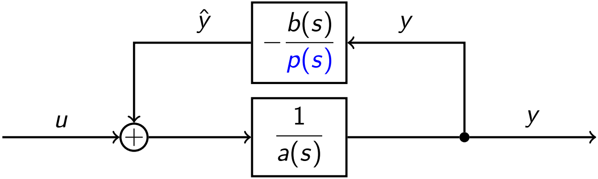


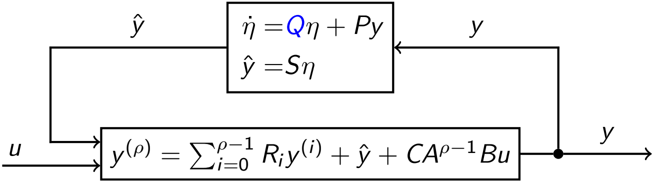
Stable zero dynamics: time domain

$$\begin{aligned}\dot{x}(t) &= A\,x(t) + B\,u(t)\\ y(t) &= C\,x(t)\end{aligned}$$

strict rel. degree $\rho \geq 1$

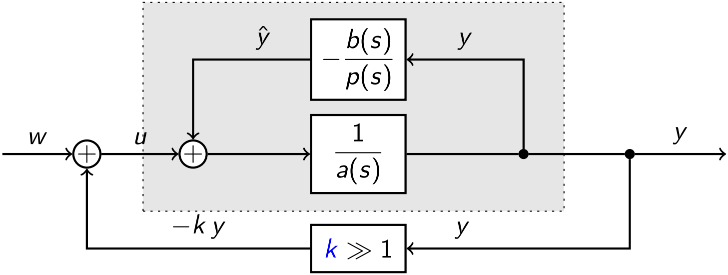
$$\mathcal{ZD}_{(A,B,C)} := \{(x,u,y) \mid (x,u,y) \text{ solves } (A,B,C) \text{ and } y \equiv 0\}$$

is **asymptotically stable** $:\Leftrightarrow$ $(x(t),u(t)) \to 0$.



Proposition: $\mathcal{ZD}_{(A,B,C)}$ is asymp. stable $\iff$ $\sigma(Q) \subset \mathbb{C}_-$

High-gain feedback: frequency domain



Suppose: $y(s) = \frac{p(s)}{q(s)}\,u(s)$ is **minimum phase**, i.e. $p(s)$ is Hurwitz,

relative degree one, i.e. $q(s) = (a_1 s - a_0)\,p(s) + b(s)$.

Then $u(s) = -k y(s) + w(s)$ yields an asymptotically stable system

$$y(s) = \frac{p(s)}{((a_1 s - a_0) + k)\,p(s) + b(s)}\;w(s)$$

High-gain feedback: time domain

$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}}$$

asymp. stable zero dynamics: $\sigma(Q) \subset \overline{\mathbb{C}}_-$

"pos." high-freq. gain $\sigma(CB) \subset \mathbb{C}_+$ $(\Rightarrow$ srdeg $1)$

High-gain feedback

$$\boxed{u(t) = -k\,y(t)}, \qquad k \gg 1$$

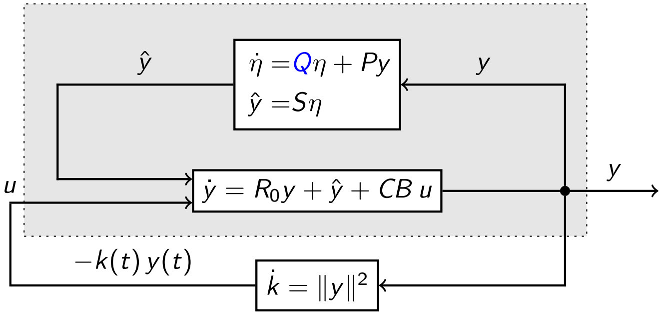
yields an exponentially stable closed-loop system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} R_0 & S \\ P & Q \end{bmatrix}\begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} CB \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} R_0 - k\,CB & S \\ P & Q \end{bmatrix}\begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix}$$

Theorem: High-gain adaptive feedback control

$$\boxed{\begin{aligned} u(t) &= -k(t)\,y(t) \\ \dot k(t) &= \|y(t)\|^2 \end{aligned}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ \eta(t) \end{pmatrix} = \begin{bmatrix} R_0 - k(t)\,CB & S \\ P & Q \end{bmatrix} \begin{pmatrix} y(t) \\ \eta(t) \end{pmatrix}$$



$$k(0) = k^0 \in \mathbb{R},\ \ x(0) = x^0 \in \mathbb{R}^n \quad\Longrightarrow\quad k(\cdot) \in L^\infty \ \wedge\ x(\cdot) \in L^\infty \ \wedge\ y(t) \to 0.$$

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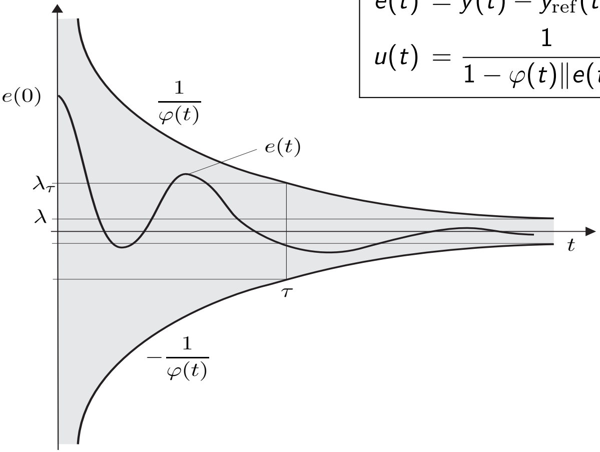
Drawbacks:

1. $k(t) \nearrow k^\infty \in \mathbb{R}$ but $k^\infty \gg 1$
2. $e(t) = y(t) - y_{\mathrm{ref}}(t) \quad \Longrightarrow \quad$ internal model required
3. $\dot{k}(t) = \|y(t) + n(t)\|,\quad n(\cdot)$ "noise" $\quad \Longrightarrow \quad k(t) \nearrow \infty$
4. $\dot{y}(t) = \varepsilon + u(t) \quad \Longrightarrow \quad k(t) \nearrow \infty \quad \wedge \quad y(t) \not\to 0$
5. $y(t) \to 0 \quad \Longrightarrow \quad$ but no transient behaviour

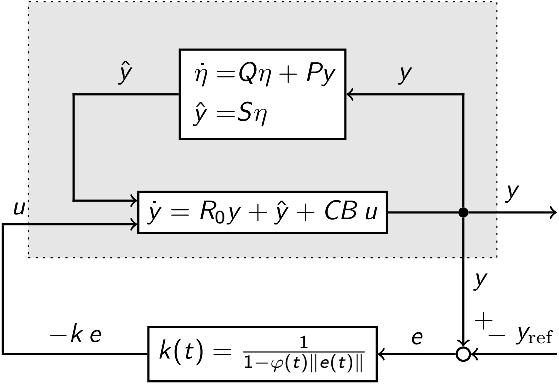
Funnel: Control objective

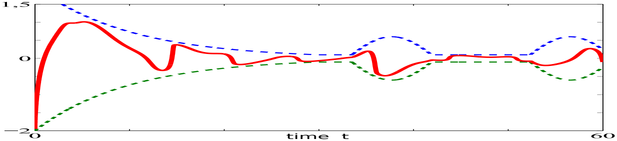


$$e(t) = y(t) - y_{\mathrm{ref}}(t)$$

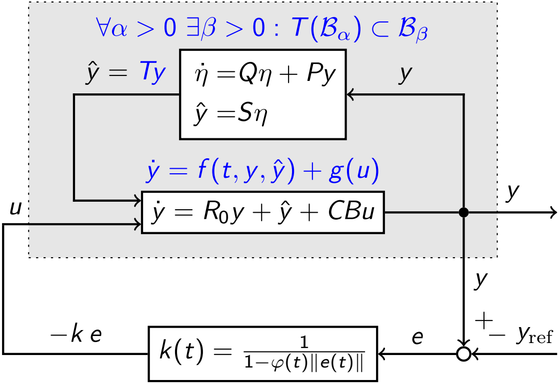
$$u(t) = \frac{1}{1 - \varphi(t)\|e(t)\|}\, e(t)$$

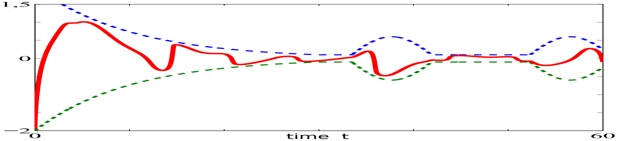
Funnel control: Theorem (linear case)





Funnel control: Theorem (nonlinear case)





Funnel control: further results

1. **Nonlinear, infinite-dimensional, hysteretic systems**
   Gene Ryan, Hartmut Logemann
2. **Electric drive systems and bioreactors**
   Hans Schuster, Christoph Hackl, Stephan Trenn
3. **Robustness in the gap metric**
   Markus Mueller, Mark French, Gene Ryan
4. **Input saturations** $u(t) = \mathrm{sat}_{[\underline{u},\overline{u}]}\big(-k(t)\,e(t)\big)$
   plus feasibility assumption
   Norman Hopfe, Gene Ryan, Stephan Trenn
5. **Higher relative degree**
   Gene Ryan, Stephan Trenn, Norman Hopfe, Markus Mueller
6. **Differential algebraic systems**
   Timo Reis, Thomas Berger
7. **Bang-bang funnel control**
   Stephan Trenn, Daniel Liberzon