Shift Spaces and the Classification of Matrices

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The Dynamics of Control

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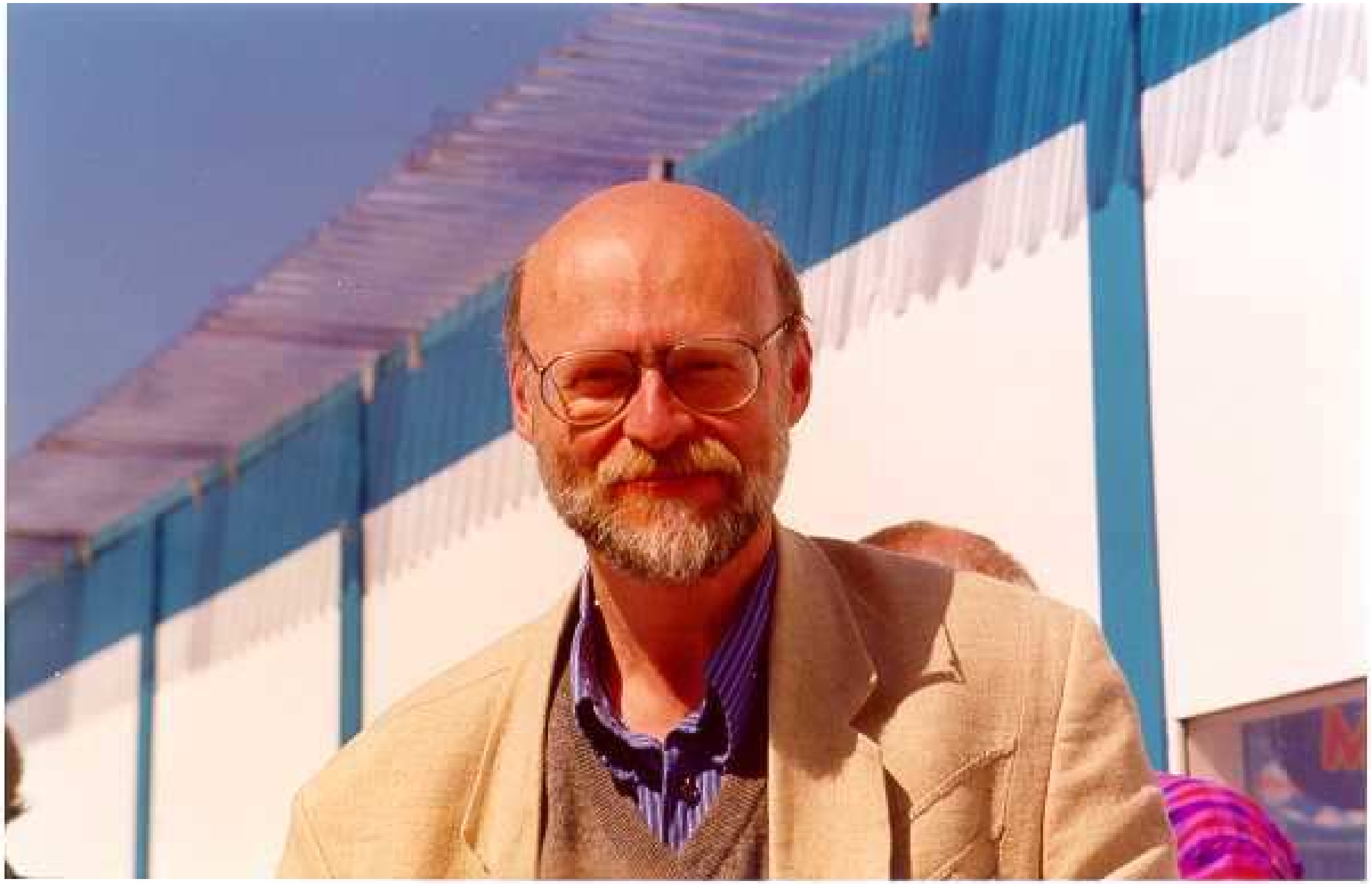
Joint Work with Paul A. Fuhrmann

HAPPY BIRTHDAY FRITZ!!



0. Motivating Examples

- Given matrices $A, B \in \mathbb{F}^{n\times n}$, find **effective algebraic criteria** for deciding similarity: $B = SAS^{-1}$.
- Jordan canonical form/rational canonical forms are not algebraically computable.
- The invariant factor algorithm is not effective.

Problem was solved 1977 by C.I. Byrnes and M.A. Gauger.

Theorem (Byrnes-Gauger, 1977) Two matrices $A, B \in \mathbb{F}^{n\times n}$ are similar if and only if

$$\det(xI - A) = \det(xI - B)$$

$$\mathrm{rk}(A \otimes I_n - I_n \otimes A) = \mathrm{rk}(B \otimes I_n - I_n \otimes B) = \mathrm{rk}(A \otimes I_n - I_n \otimes B).$$

History

Byrnes-Gauger result:

- M.A. Gauger and C.I. Byrnes. "Characteristic free, improved decidability criteria for the similarity problem", *Linear and Multilinear Algebra*, 5 (1977), 153-158.

Some Improvements and Extensions:

- J.D. Dixon. "An isomorphism criterion for modules over a principal ideal domain", *Linear and Multilinear Algebra*, 8 (1979), 69-72.
- S. Friedland. "Analytic similarity of matrices" (1980); see also his book "Matrices"; book manuscript 2009.
- W.H. Gustafson. "On matrix similarity over commutative rings", *Linear and Multilinear Algebra*, 10 (1981), 249-252.

0. Motivating Examples

- Characterization of strict equivalence for singular pencils, e.g. $(zI - A, B)$! Decidability conditions for
  - state space similarity/state feedback equivalence?
  - system immersion $\rightarrow$ model reduction?

Theorem (Byrnes/Gauger) Two single input systems $(A, b), (F, g)$ are similar iff

$$
\begin{aligned}
\det(xI - A) &= \det(xI - F) \\
\mathrm{rk}(A \otimes I_n - I_n \otimes A) &= \mathrm{rk}(F \otimes I_n - I_n \otimes F) = \mathrm{rk}(A \otimes I_n - I_n \otimes F). \\
\mathrm{rkR}(A \oplus A, b \oplus b) &= \mathrm{rkR}(F \oplus F, g \oplus g) = \mathrm{rkR}(A \oplus F, b \oplus g). \\
\dim \mathcal{O}(A \oplus A, b \oplus b) &= \dim \mathcal{O}(F \oplus F, g \oplus g) = \dim \mathcal{O}(A \oplus F, b \oplus g)
\end{aligned}
$$

0. Motivating Examples

- Given two rectangular pencils $zE - A, zF - B \in \mathbb{F}^{n\times(n+m)}$, find effective algebraic criteria for deciding strict equivalence:

$$zF - B = S(zE - A)T^{-1}.$$

- The invariant factor algorithm is not effective.
- No counterpart to Byrnes-Gauger result is known.

0. Motivating Examples

- Given two representations of an given quiver, find effective decidability conditions for checking equivalence of the representations.

1. Shift Spaces

Shift Spaces and Behaviors

Definition (J.C. Willems).
A **Shift Space/Behavior** is a linear, complete, shift-invariant subspace of $z^{-1}\mathbb{F}^m[[z^{-1}]]$.

Equivalent notions:

- **Symbolic Dynamical System**
- **Convolutional Codes:** behaviors over a finite field $\mathbb{F}$
- **Solution spaces of higher order difference eqations:**

Main Question: When are two behaviors equivalent?

1. Shift Spaces

Characterization

Theorem (J.C. Willems)
A subset $\mathcal{B} \subset z^{-1}\mathbb{F}^m[[z^{-1}]]$ is a behavior if and only if it admits a **kernel representation**, i.e. there exists a $p \times m$ polynomial matrix $P(z)$ for which

$$\mathcal{B} = \{h \in z^{-1}\mathbb{F}^m[[z^{-1}]] \mid \pi_-(Ph) = P(\sigma)h = 0\}.$$

Thus, the behaviors $\mathcal{B} \subset z^{-1}\mathbb{F}^m[[z^{-1}]]$ **are precisely the Fuhrmann rational models**

$$X^P := \{h \in z^{-1}\mathbb{F}^m[[z^{-1}]] \mid P(z)h(z) \text{ polynomial}\}.$$

1. Shift Spaces

Behaviors vs Rational Models: a la Willems & Fuhrmann

Theorem The mapping

$$M \mapsto M^{\perp} := \{f(z) \in z^{-1}\mathbb{F}^m[[z^{-1}]] \mid \mathrm{Res}_{\infty}(\mathrm{f}(\mathrm{z})^{\top}\mathrm{h}(\mathrm{z})) = 0 \ \forall \mathrm{h} \in \mathrm{M}\}$$

is a bijective, inclusion reversion correspondence between submodules of $\mathbb{F}^m[z]$ and behaviors $\mathcal{B} := M^{\perp}$ of $z^{-1}\mathbb{F}^m[[z^{-1}]]$.

1. Shift Spaces

Finite-Dimensional Behaviors

Theorem (Fuhrmann)
Equivalent are:

- A behavior $\mathcal{B}$ is a **finite-dimensional** $\mathbb{F}$-vector space.
- A behavior $\mathcal{B}$ is **autonomous**.
- $\mathcal{B}$ is a **torsion submodule** of $z^{-1}\mathbb{F}^m[[z^{-1}]]$.
- $\mathcal{B} = X^P$ for a **nonsingular** $m \times m$ polynomial matrix $P(z)$.
- $\mathcal{B}^\perp$ is a **full submodule** of $\mathbb{F}^m[z]$.

1. Shift Spaces

Thus Autonomous Behaviors are Polynomial Models!

The **polynomial model** associated with $D \in \mathbb{F}^{m\times m}[z]$ nonsingular:

$$X_D := \{f \in \mathbb{F}^m[z] \mid D(z)^{-1}f(z) \text{ strictly proper}\} \simeq \mathrm{\mathbb{F}^m[z]/D\mathbb{F}^m[z]},$$

with the module structure $z \cdot f := D\pi_-(zD^{-1}f)$.

Rational Model:

$$X^D := \{h \in z^{-1}\mathbb{F}^m[[z^{-1}]] \mid D(z)h(z) \text{ polynomial}\}.$$

They are dual friends:

$\phi_D : X_D \to X^D,\ f \mapsto D^{-1}f$, is a module isomorphism.

1. The Simplest Case:Equality of Behaviors

A Very Well Known Theorem

Let $\mathcal{B}_1, \mathcal{B}_2$ be two behaviors, defined by full row rank polynomial matrices $P_i = (T_i,\ U_i) \in \mathbb{F}^{p\times(m+p)}[z]$, with T_i invertible. Then $\mathcal{B}_1 = \mathcal{B}_2$ if and only if the transfer functions $G_i(z) = T_i(z)^{-1}U_i(z), i = 1, 2$, coincide, i.e. if and only if there exists a unimodular $U(z)$ with

$$U(z)(T_1(z),\ U_1(z)) = (T_2(z), U_2(z)).$$

Thus deciding equality of behaviors is not so much the problem!

1. Equivalence of Shift Spaces

Definition.
A **homomorphism** $Z : \mathcal{B}_1 \to \mathcal{B}_2$ between two behaviors is a continuous $\mathbb{F}$–linear map Z that intertwines the shift operators, i.e.

$$ZS^{\mathcal{B}_1} = S^{\mathcal{B}_2}Z.$$

Two behaviors are **isomorphic** or **equivalent** if there exists an invertible continuous behavior homomorphism between them.

1. Equivalence of Shift Spaces

Theorem (Fuhrmann)
Let $P(z) \in \mathbb{F}^{p \times m}[z]$ and $\bar{P}(z) \in \mathbb{F}^{\bar{p} \times \bar{m}}[z]$ be of full row rank and

$$\mathcal{B} = \mathrm{KerP}(\sigma) \quad \bar{\mathcal{B}} = \mathrm{Ker\bar{P}}(\sigma)$$

denote the associated behaviors. A map $Z : \mathcal{B}_1 \to \mathcal{B}_2$ is a **behavioral homomorphism** iff there exist $U(z) \in \mathbb{F}^{\bar{p} \times p}[z]$ and $V(z) \in \mathbb{F}^{\bar{m} \times m}[z]$ with

$$U(z)P(z) = \bar{P}(z)V(z)$$

and

$$Zh = V(\sigma)h \qquad \forall h \in \mathrm{KerP}(\sigma)$$

1. Equivalence of Shift Spaces

Theorem (Fuhrmann)
Let $P(z) \in \mathbb{F}^{p \times m}[z]$ and $\bar{P}(z) \in \mathbb{F}^{\bar{p} \times \bar{m}}[z]$ be of full row rank. Let $Z : \mathcal{B}_1 = \mathrm{KerP}(\sigma) \to \mathcal{B}_2 = \mathrm{Ker\bar{P}}(\sigma)$ be a **behavioral homomorphism**. Then

- Z is injective if and only if $P(z), V(z)$ are right coprime.
- Z is surjective if and only if $\bar{P}(z), U(z)$ are left coprime and

$$\mathrm{Ker} = \begin{bmatrix} -U(z) & \bar{P}(z) \end{bmatrix} = \mathrm{Im} \begin{bmatrix} P(z) \\ V(z). \end{bmatrix}$$

- Z is an isomorphism if and only if there exists a doubly unimodular embedding

$$\begin{bmatrix} -\bar{X}(z) & -\bar{Y}(z) \\ -U(z) & \bar{P}(z) \end{bmatrix} \begin{bmatrix} P(z) & Y(z) \\ V(z) & X(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Dynamics & Control

1. Equivalence of Shift Spaces

- Gantmacher showed, that two square matrices are similar over a field if and only if their invariant factors are equal.
- For behaviors this is wrong: Equality of the nontrivial invariant factors of the rectangular polynomial matrices is not enough to show equivalence of behaviors.

1. Equivalence of Shift Spaces

Theorem (Fuhrmann, He; 2009)
Let $P_i(z) \in \mathbb{F}[z]^{p_i \times m_i}$, $i = 1, 2$ be of full row rank, with associated behaviors $\mathcal{B}_i = X^{P_i} = \mathrm{Ker} P_i(\sigma) \subset z^{-1}\mathbb{F}[[z^{-1}]]^{m_i}$. Let the Smith form of $P_i(z)$ be $S_i(z)$, where

$$S_i(z) = \begin{pmatrix} \Delta_{r_i}^{(i)}(z) & 0 & 0 \\ 0 & I_{(p_i - r_i)\times(p_i - r_i)} & 0_{(p_i - r_i)\times(m_i - p_i)} \end{pmatrix}$$

Here $\Delta_{r_i}^{(i)} = \mathrm{diag}(d_1^{(i)}, \ldots, d_{r_i}^{(i)})$, with $d_j^{(i)}$, $j = 1, \ldots, r_i$ the nonconstant invariant factors of $P_i(z)$, i.e. $\deg d_j^{(i)} > 0$.

1. Equivalence of Shift Spaces

Then

- The behaviors X^{S_i}, $i = 1, 2$, are given by

$$X^{S_i} = \mathrm{KerS_i}(\sigma) = \left\{ \begin{pmatrix} f^{(i)} \\ 0_{(p_1 - r_1)} \\ g^{(i)} \end{pmatrix} | \mathrm{f^{(i)}} \in \mathrm{X}^{\Delta^{(i)}_{\mathrm{r_i}}}, \mathrm{g^{(i)}} \in \mathrm{z}^{-1}\mathbb{F}[[\mathrm{z}^{-1}]]^{\mathrm{m_i - p_i}}\right.$$

- We have the behavior isomorphism

$$X^{P_i} \simeq X^{S_i}.$$

1. Equivalence of Shift Spaces

- Two behaviors X^{P_1}, X^{P_2} are equivalent if and only if
  (i) $m_1 - p_1 = m_2 - p_2$ and
  (ii) V_1, V_2 have the same nonconstant invariant factors.
- We have the direct sum representation

$$X^{S_i} = X\begin{pmatrix} I_{r_i} & 0 & 0 \\ 0 & I_{p_i-r_i} & 0_{(p_i-r_i)\times(m_i-p_i)} \end{pmatrix} \oplus X\begin{pmatrix} \Delta_{r_i}^{(i)}(z) & 0 & 0 \\ 0 & I_{p_i-r_i} & 0_{(p_i-r_i)\times(m_i-p_i)} \\ 0 & 0 & I_{m_i-p_i} \end{pmatrix}.$$

  The left summand is the, uniquely determined, reachable part of the behavior X^{S_i}, whereas the second summand is autonomous.

Partially generalizes a result by Fagnani/Zampieri

Let R be P.I.D., V finitely generated R-module, endowed with discrete topology. Any closed, shift-invariant R-submodule $\mathcal{B} \subset z^{-1}V[[z^{-1}]]$ is called a behavior. The **limit rank** of $\mathcal{B}$ is

$$L(\mathcal{B}) := \lim_{n\to\infty} \frac{\mathrm{rk}_R(\mathcal{B}|_{[1,\cdots,n]})}{n}$$

1. Equivalence of Shift Spaces

Theorem (Fagnani/Zampieri (1997))
Two finite memory, controllable behaviors $\mathcal{B}_i \subset z^{-1}V_i[[z^{-1}]], i = 1, 2,$ are isomorphic if and only if the limit ranks coincide:

$$L(\mathcal{B}_1) = L(\mathcal{B}_2).$$

Lemma.

If $P \in \mathbb{F}^{p\times}[z]$ is full row rank, then the limit rank of X^P is equal to $p - m$.

2. Classification of Autonomous Behaviors

A preparatory result

Theorem (Fuhrmann, He)

The $\mathbb{F}[z]-$ tensor product of polynomial models is a polynomial model:

$$\mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_2}) \quad \simeq \quad X_{D_2} \otimes_{\mathbb{F}[z]} X_{D_1^\top} \simeq X_D$$

where $D =$ greatest common left divisor of $D_2 \otimes I$, $I \otimes D_1^\top$.

This helps us to prove (Cecioni [1908], Frobenius [1910])

Corollary

Let $d_i^{(2)}, d_j^{(1)}$ denote the invariant factors of D_2, D_1, respectively. Then

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_2}) \quad = \quad \sum_{i=1}^{p} \sum_{j=1}^{m} \deg(\gcd(d_i^{(2)}, d_j^{(1)})).$$

2. Classification of Autonomous Behaviors

Basic Dimension Inequality

For any two polynomial models we have

$$\dim \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_1}) + \dim \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_2}, X_{D_2}) \geq 2 \dim \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_2}).$$

Equality holds if and only if the modules X_{D_1}, X_{D_2} are isomorphic.

Proof uses the BG-combinatorial inequality

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_2}) = \sum_{i=1}^{p} \sum_{j=1}^{m} \deg(\gcd(d_i^{(2)}, d_j^{(1)})).$$

$$2 \sum_{i=1}^{p} \sum_{j=1}^{m} \min(n_i, m_j) \leq \sum_{i=1}^{p} \sum_{j=1}^{p} \min(n_i, n_j) + \sum_{i=1}^{m} \sum_{j=1}^{m} \min(m_i, m_j).$$

2. Classification of Autonomous Behaviors

Main Theorem A

The following are equivalent:

- The polynomial matrices $D_1(z), D_2(z)$ are equivalent, i.e. their nontrivial invariant factors are equal.
- There exists an $\mathbb{F}[z]$-isomorphism $X_{D_1} \simeq X_{D_2}$
-

$$\begin{aligned} \dim \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_1}) &= \dim \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_2}, X_{D_2}) \\ &= \dim \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_2}). \end{aligned}$$

2. Equivalence of Matrices

Let $A \in \mathbb{F}^{n \times n}$ act on $\mathbb{F}^n$ in the standard way. Let $D(z) := zI - A$. This makes $\mathbb{F}^n$ into a $\mathbb{F}[z]$-module with module isomorphism

$$\pi : X_D \to \mathbb{F}^n, \quad \sum_{j=1}^{m} v_j z^j \mapsto \sum_{j=1}^{m} A^j v_j$$

A is similar to the shift operator $S_D : X_D \to X_D$.

Corollary 1=Byrnes-Gauger Thm.

Two matrices $A, B \in \mathbb{F}^{n \times n}$ are similar if and only if

$$\mathrm{rk}(A \otimes I_n - I_n \otimes A) = \mathrm{rk}(B \otimes I_n - I_n \otimes B) = \mathrm{rk}(A \otimes I_n - I_n \otimes B).$$

2. Equivalence of Autonomous Behaviors

Corollary 2

Let $\mathcal{B}_1, \mathcal{B}_2$ denote autonomous n-dimensional behaviors and let $A \in \mathbb{F}^{n\times n}, B \in \mathbb{F}^{n\times n}$ denote the matrices, representing the shift operators on $\mathcal{B}_1, \mathcal{B}_2$, respectively. Then $\mathcal{B}_1, \mathcal{B}_2$ are equivalent if and only if

$$\mathrm{rk}(A \otimes I_n - I_n \otimes A) \quad = \quad \mathrm{rk}(B \otimes I_n - I_n \otimes B) = \mathrm{rk}(A \otimes I_n - I_n \otimes B).$$

3. Generalizations

Matrix pencils/Behavior

Theorem Two regular matrix pencils $zE - F$ and $z\overline{E} - \overline{F}$ are strict equivalent if and only if

$$\mathrm{rk}(\mathrm{F} \otimes \mathrm{E} - \mathrm{E} \otimes \mathrm{F}) = \mathrm{rk}(\overline{\mathrm{E}} \otimes \overline{\mathrm{F}} - \overline{\mathrm{F}} \otimes \overline{\mathrm{E}}) = \mathrm{rk}(\mathrm{F} \otimes \overline{\mathrm{E}} - \mathrm{E} \otimes \overline{\mathrm{F}}).$$

Theorem Consider two full row rank polynomial matrices $P_i \in \mathbb{F}^{p_i \times m_i}[z]$. The behaviors X^{P_1}, X^{P_2} are isomorphic if and only if

- $m_1 - p_1 = m_2 - p_2$.
- P_1, P_2 have identical nonconstant invariant factors.

Open problem: Effective decidability conditions?

3. Generalizations

Corollary Let G be either a compact Lie group or a complex semisimple Lie group (or a reductive Lie group).

Any finite-dimensional representations V, W of G are equivalent if and only if

$$\mathrm{dimHom_G(V,W) = dimHom_G(V,V) = dimHom_G(W,W)}.$$

Effective equivalence criteria??

3. Generalizations

Example: effective equivalence condition

Corollary Let $\rho, \tilde{\rho} : SL_2(\mathbb{C}) \to SL_n(\mathbb{C})$ be representations and

$$E := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $\rho, \tilde{\rho}$ are equivalent if and only if for $A := \rho(E), B := \tilde{\rho}(E)$:

$$\mathrm{rank}(\mathrm{I_n} \otimes \mathrm{A} - \mathrm{A} \otimes \mathrm{I_n}) = \mathrm{rank}(\mathrm{I_n} \otimes \mathrm{B} - \mathrm{B} \otimes \mathrm{I_n}) = \mathrm{rank}(\mathrm{I_n} \otimes \mathrm{A} - \mathrm{B} \otimes \mathrm{I_n}).$$

Proof: uses Kostant-Segikuchi correspondence for $\mathfrak{sl}_2(\mathbb{C})$
Open problem for other groups

4. Open Problems and Questions

- Decidability conditions for isomorphy of behaviors.
- Equivalence for parametric systems: systems over rings
  - Lots of old work by Byrnes, Friedland, Hazewinkel, Sontag, Martin/Khadr ...
- Opens new numerical approach to Sylvester/Lyapunov equations, ...
  - Extensions of work by Edelman, Kagstrom, ...
- Equivalence of periodic behaviors; behaviors with symmetries
  - Floquet theory; Lifting procedures (Van Dooren; Verriest, ...)