Zubov's Method for Differential Games

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Joint work with Oana Silvia Serea,
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International Workshop "The Dynamics of Control"
Irsee, 1st–3rd October, 2010

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Happy Birthday Fritz!

Introduction: The Uncontrolled Case

Consider the autonomous ODE

$$\dot{x}(t) = f(x(t)), \qquad x \in \mathbb{R}^d$$

with solutions $\Phi(t, x_0)$ and locally exponentially stable equilibrium $x^* \in \mathbb{R}^d$

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i.e., there exists a neighborhood $\mathcal{N}$ of $x^* = 0$ and constants $c, \sigma > 0$, such that for all $x_0 \in \mathcal{N}$:

$$\|\Phi(t, x_0)\| \le ce^{-\sigma t}\|x_0\|$$

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$$\|\Phi(t, x_0)\| \leq c e^{-\sigma t} \|x_0\|$$

Problem: What is the domain of attraction

$$\mathcal{D} := \{x \in \mathbb{R}^d \,|\, \Phi(t, x) \to x^* = 0\} \ ?$$

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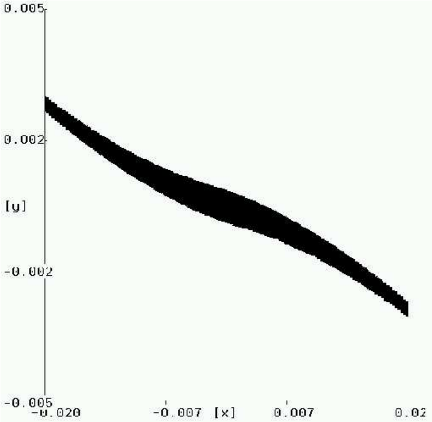
Example for a Domain of Attraction

Fluid Dynamics: Explanation of the difference between linear stability and experimental instability for large Reynolds numbers [Trefethen et al., Science, 1993]



Zubov's Equation [1964]

For a continuous function $h : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ with $h(x) = 0 \Leftrightarrow x = x^*$ consider the PDE "Zubov's Equation"

$$Dw(x) \cdot f(x) = -h(x)(1 - w(x))$$

with $w : \mathbb{R}^d \to \mathbb{R}$ and boundary condition $w(x^*) = 0$

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Then: under suitable conditions on h this equation has a unique solution $w : \mathbb{R}^d \to [0, 1]$ with

$$w(x) = 0 \;\Leftrightarrow\; x = x^*$$

and $\mathcal{D}$ satisfies the level set characterization

$$\mathcal{D} = w^{-1}([0, 1)) := \{x \in \mathbb{R}^d \,|\, w(x) \in [0, 1)\}$$

Example

$$\dot{x}_1(t) = -x_1(t) + x_1(t)^3, \quad \dot{x}_2(t) = -x_2(t) + x_2(t)^3$$

$$\mathcal{D} = [-1, 1]^2$$

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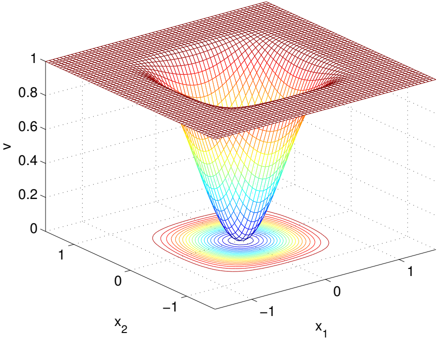
$$\mathcal{D} = [-1,1]^2, \qquad h(x) = 5\|x\|^2$$



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Integral Equation

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Integration of Zubov's equation and subsequent integration by parts yields the integral equation

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Thus:

$$\Phi(t,x) \to x^* \quad \Leftrightarrow \quad \int_0^\infty h(\Phi(t,x))dt < \infty \quad \Leftrightarrow \quad w(x) < 1$$

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Zubov's Equation — Discussion

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- periodic orbits [Aulbach '83]
- perturbed systems (deterministic: [Camilli, Gr., Wirth '01], stochastic: [Camilli, Loreti '06; Camilli, Gr. '03])
- control systems (deterministic: [Sontag '83, Camilli, Gr., Wirth '08], stochastic: [Camilli, Cesaroni, Gr., Wirth '06])

Control and Perturbation

In this talk we consider generalizations of this method for controlled and deterministically perturbed systems

$$\dot{x}(t) = f(x(t), u(t), v(t))$$

with $x(t) \in \mathbb{R}^d$
$u \in \mathcal{U} = \{u : [0, \infty) \to U, \text{ measurable}\}$
$v \in \mathcal{V} = \{v : [0, \infty) \to V, \text{ measurable}\}$
$U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^l$ compact

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Problem: stabilization under uncertainty

$u \mathrel{\hat{=}}$ control, trying to achieve $\Phi(t, x_0, u, v) \to x^*$

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(convergence to $x^* = 0$ can be generalized to arbitrary compact sets)

Extension of Integral Equation

Recall: Zubov's method relies on the integral equation

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Local exponential controllability and an appropriate choice of $h : \mathbb{R}^d \times U \times V \to \mathbb{R}_0^+$ ensure

$$J(x,u,v) < 1 \quad \Leftrightarrow \quad \Phi(t,x,u,v) \to 0$$

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$\rightsquigarrow$ zero sum differential game (min-max problem)

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Information Exchange between u and v

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General question for differential games: does the "infinitesimal" advantage make a difference?

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Formalization of the Information Structure

We formalize the last two cases by defining the set Δ of nonanticipative strategies for the perturbation as the set of maps $\beta : \mathcal{U} \to \mathcal{V}$ with the following property for all u_1, $u_2 \in \mathcal{U}$ and all $s > 0$:

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$\rightsquigarrow$ upper value: $w^+(x) := \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} J(x, u, \beta(u))$

lower value: $w^-(x) := \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} J(x, \alpha(v), v)$

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Keep in mind: the strategy player has an infinitesimal advantage

Domains of Controllability

We need two different domains of controllability

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⇝ upper domain of uniform asymptotic controllability

$$\mathcal{D}^+ = \left\{ x \in \mathbb{R}^d \;\middle|\; \begin{array}{l} \text{there exists } \theta(t) \to 0 \text{ such that} \\ \text{for each } \beta \in \Delta \text{ there exists} \\ u \in \mathcal{U} \text{ with } \|\Phi(t,x,u,\beta(u))\| \le \theta(t) \end{array} \right\}$$

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Local exponential controllability is defined analogously with $\theta(t) = ce^{-\sigma t}\|x\|$

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$$\mathcal{D}^+ = \left\{ x \in \mathbb{R}^d \;\middle|\; \begin{array}{l} \text{there exists } \theta(t) \to 0 \text{ such that} \\ \text{for each } \beta \in \Delta \text{ there exists} \\ u \in \mathcal{U} \text{ with } \|\Phi(t,x,u,\beta(u))\| \le \theta(t) \end{array} \right\}$$

$\rightsquigarrow$ lower domain of uniform asymptotic controllability

$$\mathcal{D}^- = \left\{ x \in \mathbb{R}^d \;\middle|\; \begin{array}{l} \text{there exists } \theta(t) \to 0 \text{ and } \alpha \in \Gamma \\ \text{such that for each } v \in \mathcal{V} \text{ the inequality} \\ \|\Phi(t,x,\alpha(v),v)\| \le \theta(t) \text{ holds} \end{array} \right\}$$

Local exponential controllability is defined analogously with
$\theta(t) = ce^{-\sigma t}\|x\|$ (can be generalized to uniform convergence)

Example

Can the upper and lower domain be different?

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Formal Derivation of Zubov's Equation

$$w^+(x) := \sup_{\beta\in\Delta} \inf_{u\in\mathcal{U}} \left\{ 1 - e^{-\int_0^\infty h(\Phi(t,x,u,v),u(t),v(t))dt} \right\}$$

satisfies for all $T > 0$ the optimality principle

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Division by $-T$ and passing to the limit for $T \to 0$ yields the Hamilton–Jacobi–Isaacs equation

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"generalized Zubov Equation"

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Likewise, w^- formally satisfies the generalized Zubov equation

$$H^-(x, w^-(x), Dw^-(x)) = 0$$

with Hamiltonian

$$H^-(x, w, p) = \inf_{v \in V} \sup_{u \in U} \{-p \cdot f(x, u, v) - h(x, u, v)(1 - w)\}$$

Nonsmooth Solution

Problem: w^+ and w^- from the integral equations are typically nonsmooth

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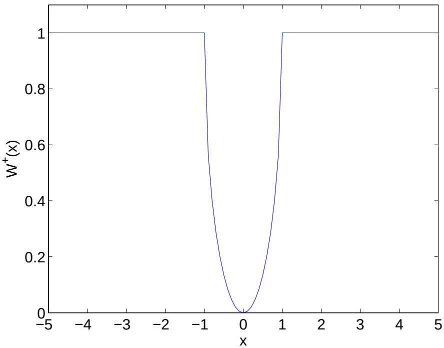
Example: w^+ for $\dot{x}(t) = -x(t) + u(t)v(t)x(t)^3$, with $U = V = \{-1, 1\}$, $h(x,u,v) = x^2$



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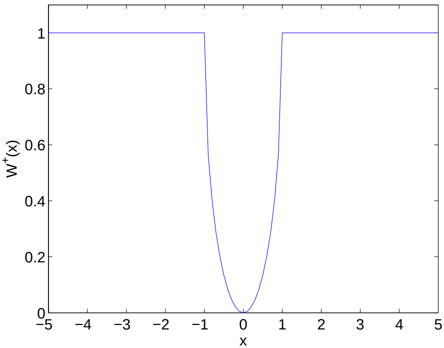
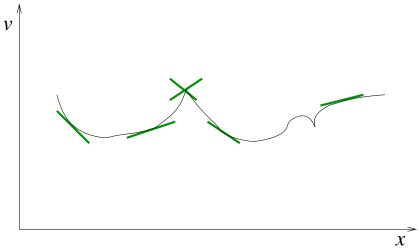


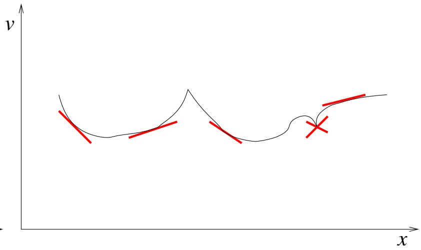
$\rightsquigarrow$ viscosity solution

Viscosity Solution

Super– and subdifferential:



$D^{+}v(x)$



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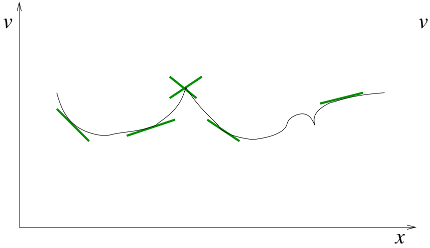

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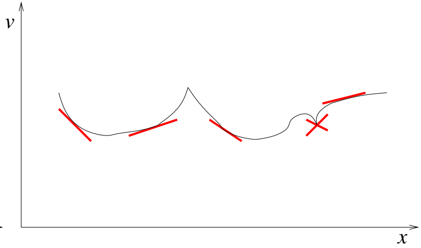

$D^-v(x)$

w viscosity supersolution: $H(x, w(x), p) \geq 0$ for all $p \in D^-w(x)$

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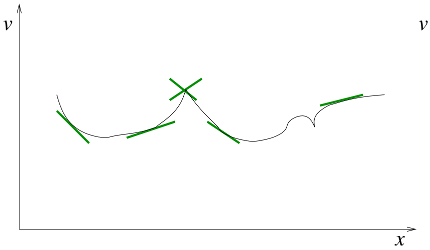

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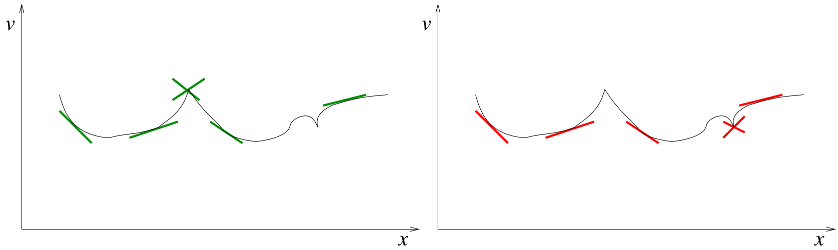

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w viscosity solution, if both holds [Crandall, Lions 82]

Existence and Uniqueness

With this solution concept and with the help of "sub– and superoptimality principles" for viscosity super– and subsolutions [Soravia 95] we arrive at the following Theorem:

w^+ is the unique continuous viscosity solution of the equation

$$\sup_{u \in U} \inf_{v \in V} \{-p \cdot f(x,u,v) - h(x,u,v)(1-w)\} = 0$$

with $w^+(0) = 0$

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with $w^-(0)=0$

Furthermore, the characterizations $\mathcal{D}^+=(w^+)^{-1}([0,1))$ and $\mathcal{D}^-=(w^-)^{-1}([0,1))$ hold

Example

Consider the example from before

$$\dot{x}(t) = -x(t) + u(t)v(t)x(t)^3, \quad x(t) \in \mathbb{R}$$

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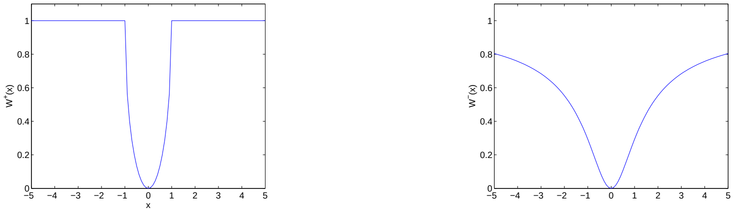
For $h(x, u, v) = x^2$ we can compute explicitly

$$w^+(x) = \begin{cases} 1 - \sqrt{1 - x^2}, & |x| < 1 \\ 1, & |x| \geq 1 \end{cases} \qquad w^-(x) = \frac{\sqrt{1 + x^2} - 1}{\sqrt{1 + x^2}}$$



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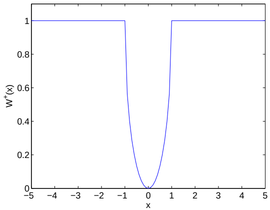

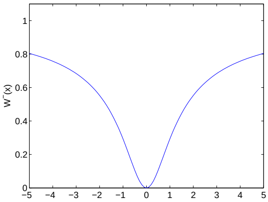

This confirms $\mathcal{D}^+ = (-1, 1)$ and $\mathcal{D}^- = (-\infty, \infty)$

When does $\mathcal{D}^+ = \mathcal{D}^-$ hold?

When does playing strategies yield no advantage?

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Recall:

$$H^+(x, w, p) = \sup_{u \in U} \inf_{v \in V} \{-p \cdot f(x, u, v) - h(x, u, v)(1 - w)\}$$

$$H^-(x, w, p) = \inf_{v \in V} \sup_{u \in U} \{-p \cdot f(x, u, v) - h(x, u, v)(1 - w)\}$$

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For the special case of $h(x,u,v) = h(x)$ we get

$$\begin{aligned} H^+(x,w,p) &= H^-(x,w,p) \\ \Leftrightarrow \quad \sup_{u\in U} \inf_{v\in V} \{-p \cdot f(x,u,v)\} &= \inf_{v\in V} \sup_{u\in U} \{-p \cdot f(x,u,v)\} \end{aligned}$$

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Theorem: Isaacs' condition implies $\mathcal{D}^+ = \mathcal{D}^-$

This theorem extends a well known result from capture basins in finite time pursuit evasion games to domains of controllability of asymptotically controllable sets

Example

In our example

$$\dot{x}(t) = -x(t) + u(t)v(t)x(t)^3, \quad x(t) \in \mathbb{R}$$

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$\rightsquigarrow$ Isaacs' condition must be violated

Indeed, for $p = 1$ and $x = 1$ we have

$$p \cdot f(x, u, v) = -1 + uv$$

and thus

$$\sup_{u \in U} \inf_{v \in V} \{-p \cdot f(x, u, v)\} = \sup_{u \in U} \inf_{v \in V} \{1 - uv\} = 0$$

but

$$\inf_{v \in V} \sup_{u \in U} \{-p \cdot f(x, u, v)\} = \inf_{v \in V} \sup_{u \in U} \{1 - uv\} = 2$$

Conclusions

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- Zubov's method gives a characterization of the domain of asymptotic controllability via the level set of a solution of a first order PDE
- Using viscosity solutions, the method can be extended to a differential game setting
- Upper and lower value w^+ and w^- and the respective domains of controllability $\mathcal{D}^+$ and $\mathcal{D}^-$ are defined and analyzed seperately
- Under the well known Isaacs condition the upper and lower domains of controllability $\mathcal{D}^+$ and $\mathcal{D}^-$ coincide