Some questions in non-autonomous linear control processes: the infinite-horizon H^∞ control problem.

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joint work with

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- Non-autonomous linear control processes.
- Non-autonomous linear Hamiltonian systems.
- Rotation number.
- Exponential dichotomy.
- Non-autonomous linear H^∞ control with infinite horizon.
- Riccati equation $\leftrightarrow$ linear Hamiltonian system
- Critical attenuation value $\longleftarrow$ Exponential dichotomy
rotation number

$$
\begin{aligned}
x' &= A(t)x + B(t)u + D(t)w \qquad (1)\\
x(0) &= x_0
\end{aligned}
$$

$x \in \mathbb{R}^n$ state vector

$u \in \mathbb{R}^m$ control vector

$w \in \mathbb{R}^l$ disturbance (generic L^2)

A, B, D uniformly bounded and uniformly continuous functions of $t \in \mathbb{R}$ with values in the appropriate sets of matrices.

For each $\gamma > 0$ consider the functional

$$L_\gamma(u,w) = \int_0^\infty \{< Q(t)x(t), x(t) > + \\ + < u(t), u(t) > -\gamma^{-2} < w(t), w(t) >\} \, dt \tag{2}$$

$Q(t)$ symmetric $n \times n$ real matrix

$Q(t) \geq 0 \ \forall t \in \mathbb{R}$

$\forall \gamma > 0 \Rightarrow u = -B^t(t) m_\gamma(t) x$ with $m_\gamma(\cdot)$ symmetric, positive definite $n \times n$ matrix s.t.

$i)$ If $w = 0$ system (1) is stable

$ii)$ $L_\gamma(u,w) \leq < m_\gamma(0) x_0, x_0 >$

minimal attenuation value

$\gamma^* = \inf\{\gamma > 0 \mid$ exists a linear feedback control u s.t. $i)$ and $ii)$ hold$\}$.

Matrix Riccati equation

$$m' + A^t m + mA - m[BB^t - \gamma^{-2} DD^t]m + Q = 0 \tag{3}$$

If for some $\gamma > 0$, $m(t)$ is bounded (non-conjugate) on all of $\mathbb{R}$, then the H^∞-control problem admits a solution $u = -B^t(t)m(t)x$.

Riccati eq.(3) $\leftrightarrow$ linear non-autonomous Hamiltonian system (4)

$$z' = \begin{pmatrix} A(t) & -[B(t)B^t(t) - \gamma^{-2} D(t)D^t(t)] \\ -Q(t) & -A^t(t) \end{pmatrix} z \tag{4}$$

where $z = \binom{x}{y} \in \mathbb{R}^{2n}$

Bebutov or translation flow.

$\mathbb{M}_{r,s}$ set of $r \times s$ real matrices $(1 \leq r, s < \infty)$.

$\mathcal{G}_{r,s} = \{G : \mathbb{R} \to \mathbb{M}_{r,s} \mid G$ is uniformly bounded and uniformly continuous$\}$.

$\mathcal{G}_{r,s}$ topology of uniform convergence on compact subsets of $\mathbb{R}$.

$\{\tau_t \mid t \in \mathbb{R}\}$ Bebutov flow on $\mathcal{G}_{r,s}$:

$$G \in \mathcal{G}_{r,s}, \;\; \tau_t(G)(\cdot) = G(\cdot + t) \;\; t \in \mathbb{R}$$

$(\mathcal{G}_{r,s}, \{\tau_t\})$ flow

$\mathcal{G} = \mathcal{G}_{n,n} \times \mathcal{G}_{n,m} \times \mathcal{G}_{n,l} \times \mathcal{G}_{n,n}$

$\xi_0 = (A, B, D, Q) \in \mathcal{G}$

$\Xi = cls\{\tau_t(\xi_0) \mid t \in \mathbb{R}\}$ compact, translation-invariant subset of $\mathcal{G}$

$\xi \in \Xi \Rightarrow (A_\xi, B_\xi, D_\xi, Q_\xi)$

$\mathcal{A} : \mathcal{G} \to \mathbb{R} : (a, b, d, q) \to a(0)$

Then $A_\xi(t) = \mathcal{A}(\tau_t(\xi))$. Similarly we obtain B_ξ, D_ξ and Q_ξ with $\mathcal{B}$, $\mathcal{G}$ and $\mathcal{Q}$.

For each $\xi \in \Xi$ and $\gamma > 0$, we consider

$$x' = A_\xi(t)x + B_\xi(t)u + D_\xi(t)w \qquad (1_\xi)$$
$$x(0) = x_0$$

and the functional

$$L_{\gamma.\xi}(u, w) = \int_0^\infty \{< Q_\xi(t)x(t), x(t) > + |u(t)|^2 - \gamma^2 |w(t)|^2\}\, dt$$

Riccati equation

$$m' + A_\xi^t m + m A_\xi - m[B_\xi B_\xi^t - \gamma^{-2} D_\xi D_\xi^t] m + Q_\xi = 0 \qquad (3_\xi)$$

and the related Hamiltonian systems

$$z' = \begin{pmatrix} A_\xi & -[B_\xi B_\xi^t - \gamma^{-2} D_\xi D_\xi^t] \\ -Q_\xi & -A_\xi^t \end{pmatrix} z, \;\; z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad (4_\xi)$$

$z_1(t), \ldots, z_n(t)$ n linearly independent solutions of (4_ξ)

Write $(z_1(t), \ldots, z_n(t))$ $2n \times n$ matrix as $\binom{X(t)}{Y(t)}$

$X(t)$ invertible on $I \subset \mathbb{R} \Rightarrow m(t) = Y(t)X(t)^{-1}$ solution of the Riccati equation (3_ξ) on I.

Exponential dichotomy (E.D.)

Hyperbolic splitting of the space $\Xi \times \mathbb{R}^{2n}$.

The family of differential equations (4_ξ) admits an exponential dichotomy over Ξ if there are positive constants k, β and a continuous function $P : \Xi \to \mathcal{P} : \xi \to P_\xi$ such that

$$\|\Phi_\xi(t) P_\xi \Phi_\xi(s)^{-1}\| \le K e^{-\beta(t-s)} \;, t \ge s$$
$$\|\Phi_\xi(t)(I - P_\xi)\Phi_\xi(s)^{-1}\| \le k e^{\beta(t-s)} \;, t \le s$$

where $\mathcal{P}$ is the family of all linear projections $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $\Phi_\xi(t)$ is the fundamental matrix solution of (4_ξ) for each $\xi \in \Xi$.

$\lambda \subset \mathbb{R}^{2n}$ n dim.

λ is a Lagrange plane if $< x, Jy >= 0$ $\forall x, y \in \lambda$. $J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$.

$\Lambda \subset \mathbb{R}^{2n}$ Lagrange subspaces.

$\frac{n(n+1)}{2}$-dimensional real analytic manifold.

$\lambda_h = \begin{bmatrix} I_n \\ 0_n \end{bmatrix}$ horizontal Lagrange plane

$\lambda_v = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}$ vertical Lagrange plane

If $\lambda \in \Lambda$ is transversal to λ_v i.e. $\lambda \cap \lambda_v = 0 \Rightarrow \lambda = \begin{bmatrix} I_n \\ m \end{bmatrix}$ where m is a $n \times n$ symmetric real matrix.

If in addition λ transverse to $\lambda_h \Rightarrow$ $det\, m \neq 0$. m parametrizes λ.

$$C_v = \{\lambda \in \Lambda \mid \lambda \text{ is } \underline{\text{not}} \text{ transversal to } \lambda_v\}$$

C_v vertical Maslov cycle

$$C_h = \{\lambda \in \Lambda \mid \lambda \text{ is } \underline{\text{not}} \text{ transversal to } \lambda_h\}$$

C_h horizontal Maslov cycle.

μ fixed ergodic measure on Ξ

$C_v \subset \Lambda$ 2-sided in Λ.

$\Lambda \setminus C_v$ simply connected.

Oriented intersection index $i(c)$ of each continuous closed curve $c : [0, T] \to \Lambda$ with C_v whenever $c(0)$ and $c(T)$ lie off C_v.

$\xi \in \Xi$, $T > 0$ $\lambda \in \Lambda$ transverse to $\lambda_v = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}$.

$c_T(t) = \Phi_\xi(t) \cdot \lambda \Rightarrow c_T : [0, T] \to \Lambda$

continuous closed curve in Λ.

n_T intersection index of c_T with C_v.

rotation number

$$\alpha(\mu) = -\lim_{T \to \infty} \pi \frac{n_T}{T}$$

The limit exists and is independent on the choice of λ, ξ $(\mu{-}a.e.)$.

Rotation number $\leftrightarrow$ E.D.

Atkinson condition.

$\Gamma : \Xi \to \mathbb{M}_{2n,2n}$ symmetric and $\Gamma(\xi) \geq 0$ for all $\xi \in \Xi$.

$\Gamma_\xi(t) = \Gamma(\tau_t(\xi))$ $(\xi \in \Xi, t \in \mathbb{R})$.

(4_ξ) satisfy an Atkinson condition with respect to Γ if each minimal subset $M \subset \Xi$ contains a point p such that

$$\int_{-\infty}^{\infty} |\Gamma_p(t)\Phi_p(t)|^2 > 0.$$

Using control-theoretic arguments (Johnson-Nerurkar 1994) it can be shown that the Atkinson condition implies $\exists T > 0, \delta > 0$ s.t.

$$\int_0^T |\Gamma_\xi(t)\Phi_\xi(t)|^2 \, dt \geq \delta I$$

$\forall \xi \in \Xi$.

Theorem (J-Nerurkar 1994) $\Xi = supp\,\mu$

Consider the Atkinson-type spectral problem

$$z' = \left[\begin{pmatrix} A_\xi & -B_\xi B_\xi^t \\ -Q_\xi & -A_\xi^t \end{pmatrix} + \eta J^{-1} \Gamma_\xi \right] z, \quad (\xi \in \Xi) \tag{5_ξ}$$

where $\eta \in \mathbb{C}$. For each $\eta \in \mathbb{R}$, let $\alpha = \alpha(\eta)$ be the rotation number of (5_ξ) with respect to μ. Suppose that the Atkinson condition hold. Suppose that $\alpha(\mu)$ is constant on some open interval $I \subset \mathbb{R}$. Then for each $\eta \in I$, the family admits an exponential dichotomy on Ξ.

For each $\xi \in \Xi$, $x_0 \in \mathbb{R}^n$ and $\gamma > 0$, we consider the system (1_ξ)

$$x' = A_\xi(t)x + B_\xi(t)u + D_\xi(t)w$$

and the functional

$$L_{\gamma,\xi}(u,w) = \int_0^\infty \{< Q_\xi(t)x(t), x(t) > + |u(t)|^2 - \gamma^2 |w(t)|^2\}\, dt.$$

Find $\gamma > 0 \Rightarrow$ linear feedback control u with $u = -B_\xi(t)^t m_{\gamma,\xi}(t)x$ which stabilizes the system

$$x' = A_\xi(t)x + B_\xi(t)u \ (w = 0, \text{ cond. } i))$$

and for which the inequality

$$L_{\gamma,\xi}(u,w) \leq < m_{\gamma,\xi}(0)x_0, x_0 > \ (\text{cond. } ii))$$

holds (with $w \in L^2([0,\infty), \mathbb{R}^l), \xi \in \Xi, x_0 \in \mathbb{R}^n$).

Controllability Conditions.

Hypothesis 1. Each minimal subset $M \subset \Xi$ contains a point p such that the control system

$$y' = -A_p^t(t)y + Q_p(t)v$$

is null-controllable.

Hypothesis 2. Each minimal subset $M \subset \Xi$ contains a point p such that the control system

$$x' = A_p(t)x + B_p(t)u$$

is null controllable.

Definition We define the critical attenuation value γ^* for the family (4_ξ) as

$$\gamma^* = \text{inf}\{\bar{\gamma} \mid \text{ for all } \gamma \geq \bar{\gamma}, \text{ eqns. } (4_\xi) \text{ admit an E.D. over } \Xi, \text{ and } \lambda(\xi) \text{ is transverse to } \lambda_v \text{ for all } \xi \in \Xi\}$$

Theorem 1. (Colonius, F., Johnson)

Consider the family of H^∞ control problems defined by equations (1_ξ) and the functional $L_{\gamma,\xi}$ $(\xi \in \Xi)$. Suppose that the Hypotheses 1. and 2. are valid. Let γ^* be the critical attenuation value for the family (4_ξ). Suppose that $\gamma > \gamma^*$. Then for each $\xi \in \Xi$, there is a linear feedback control $u = -B^t_\xi(t) m_\xi(t) x$ such that the system

$$x' = [A_\xi(t) - B_\xi(t) B^t_\xi(t) m_\xi(t)]x$$

is uniformly exponentially stable. Moreover for all $x_0 \in \mathbb{R}^n$ and all $w \in L^2([0,\infty), \mathbb{R}^l)$ one has

$$L_{\gamma,\xi}(u,w) \leq < m_\xi(0) x_0, x_0 > .$$

$m_\xi(t)$ is positive definite for all $\xi \in \Xi$ and $t \in \mathbb{R}$.

Hypothesis 3. Assume that the Atkinson condition holds for equations (4_ξ) with

$$\Gamma_\xi(t) = \begin{pmatrix} 0_n & 0_n \\ 0_n & D_\xi(t) D_\xi^t(t) \end{pmatrix}.$$

Definition Set

$$\gamma_l = \inf\{\bar{\gamma} \mid \text{ equations } (4_\xi) \text{ admit an E.D. over } \Xi \text{ for all } \gamma > \bar{\gamma}\}.$$

Theorem 2. (Colonius, F., Johnson)

Consider the family of H^∞ control problems defined by (1_ξ) and the functional $L_{\gamma,\xi}$ ($\xi \in \Xi$). Suppose that the controllability Hypotheses 1., 2. and 3. are valid. Let γ^* be the critical attenuation value for the family (4_ξ), and let γ_l as in the Definition. Then $\gamma_l > 0$ and $\gamma^* \geq \gamma_l$.

Remark

Two possibilities: $\gamma^* > \gamma_l$ and $\gamma^* \leq \gamma_l$.

$\gamma^* > \gamma_l \Rightarrow \exists\, \xi^* \in \Xi$ such that $\lambda_{\gamma^*}(\xi^*) \in C_v$.

If $\gamma \leq \gamma^*$ and $\xi \in \Xi$ with $\{\tau_t(\xi) \mid t \geq 0\}$ dense in Ξ, then there is no linear feedback control u for which the dissipation inequality $ii)$ holds.

$\gamma = \gamma^*$ more interesting. The attenuation problem may or may not be solvable at $\gamma = \gamma^*$ when $\gamma^* = \gamma_l$. We gave an example where $\gamma^* = \gamma_l$ and the attenuation problem admits multiple solutions at $\gamma = \gamma^*$: there are distinct feedback controls u_1 and u_2 verifying $i)$ and $ii)$.

Non linear case:

$$\begin{cases} x' = A(t)x + B(t)u + D(t)w + N(t,x,u,w) \\ x(0) = x_0 \end{cases}$$

The function N is of second order in (x, u, w) unif. in $t \in \mathbb{R}$.

$N(t,x,u,w) = N_1(t,x,u) + N_2(t,x,u,w)w$

$N_1 = O\left((|x| + |u|)^2\right)$ as $(x, u) \to (0, 0)$ unif. in $t \in \mathbb{R}$.

$N_2(t,x,u,w) = O(|x| + |u|)$ unif. in $t \in \mathbb{R}$ and $|w| \leq 1$.

Fix $\gamma > \gamma^*$ and $u = -B^t(t)m_\gamma(t)x = F(t)x$ with m_γ bounded solution of the Riccati equation.

$$(m' + A^t m + mA - m[BB^t - \gamma^{-2}DD^t]m + Q = 0)$$

Write

$$N_1(t, x, F(t)x) = n_1(t, x)$$

and

$$N_2(t, x, F(t)x, w) = n_2(t, x, w)$$

Hypotheses on the nonlinear part

Theorem 3.(F-Johnson)

Assume that $Q(t)$ is strictly positive definite for all $t \in \mathbb{R}$, and the derivative $\frac{d}{dt}Q(t)$ is uniformly bounded and uniformly continuous on $\mathbb{R}$. Let $\varepsilon > 0$ be given. Then we have:

1) There exists $\eta_1 = \eta_1(\gamma, \varepsilon) > 0$ such that, if $|x_0| \leq \eta_1$, then the solution of

$$x' = [A(t)x + B(t)F(t)]x + n_1(t,x)$$
$$x(0) = x_0$$

is exponentially stable [w=0 cond. i)]

2) There exists $\eta = \eta(\gamma, \varepsilon) > 0$ such that, if $|x_0| \leq \eta$ and $|w|_\infty \leq \eta$, then the corresponding solution of (8) has the property that

$$\int_0^\infty \{< Q(t)x(t), x(t) > +|u(t)|^2\}\, dt \leq$$
$$\leq (1+\varepsilon)\gamma^2 \int_0^\infty |w(t)|^2\, dt +$$
$$+ (1+\varepsilon) < m(0)x_0, x_0 > \qquad \text{[cond. } ii\text{)]}$$

Linear case

$$L_\gamma(u,w) \leq < m(0)x_0, x_{0>}$$

Nonlinear case (locally)

$$L_\gamma(u,w) \leq \varepsilon\gamma^2 \int_0^\infty |w(t)|^2\,dt + (1+\varepsilon) < m(0)x_0, x_0 >$$

where

$$L_\gamma(u,w) = \int_0^\infty \{< Q(t)x(t), x(t) > + |u(t)|^2 + \\ + \gamma^{-2}|w(t)|^2\}\,dt$$

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